

On the Fredholm and Unique Solvability of Nonlocal Elliptic Problems in Multidimensional Domains

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Abstract

We consider elliptic equations of order $2m$ in a bounded domain $Q \subset \mathbb{R}^n$ with nonlocal boundary-value conditions connecting the values of a solution and its derivatives on $(n-1)$ -dimensional smooth manifolds Γ_i with the values on manifolds $\omega_i(\Gamma_i)$, where $\bigcup_i \overline{\Gamma_i} = \partial Q$ is a boundary of Q and ω_i are C^∞ diffeomorphisms. By proving a priori estimates for solutions and constructing a right regularizer, we show the Fredholm solvability in weighted space. For nonlocal elliptic problems with a parameter, we prove the unique solvability.

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Introduction

We consider elliptic equations of order $2m$ in a bounded domain $Q \subset \mathbb{R}^n$ with nonlocal boundary-value conditions connecting the values of a solution and its derivatives on $(n-1)$ -dimensional smooth manifolds Γ_i with the values on manifolds $\omega_i(\Gamma_i)$, where $\bigcup_i \overline{\Gamma_i} = \partial Q$ is a boundary of Q and ω_i are C^∞ diffeomorphisms. The presence of nonlocal terms leads to appearing power-law singularities of solutions and their derivatives at the points of the set $\mathcal{K}_1 = \bigcup_i (\overline{\Gamma_i} \setminus \Gamma_i)$ (which is called the *set of conjugation points*). Therefore, nonlocal elliptic problems are naturally studied in weighted spaces $H_a^{l+2m}(Q)$, $a \in \mathbb{R}$, $l \geq 0$ is an integer (see definition (1.5)), originally introduced in the theory of elliptic problems in nonsmooth domains [15]. Because of the transformations ω_i occurring in nonlocal terms, the points of the set \mathcal{K}_1 turn out to be connected with the points of the set

$$\left\{ \bigcup_i \omega_i(\mathcal{K}_1) \right\} \cup \left\{ \bigcup_{i,j} \omega_j(\omega_i(\mathcal{K}_1) \cap \Gamma_j) \right\}.$$

The latter points belong to Q or ∂Q . Therefore, we must consider certain consistency conditions at the points of the set

$$\mathcal{K} = \mathcal{K}_1 \cup \left\{ \bigcup_i \omega_i(\mathcal{K}_1) \right\} \cup \left\{ \bigcup_{i,j} \omega_j(\omega_i(\mathcal{K}_1) \cap \Gamma_j) \right\}.$$

The following two approaches are possible. First, one can consider all the points of the set \mathcal{K} as singular points in the definition of weighted spaces, which allows one to study nonlocal elliptic problems for any value of the parameter $a \in \mathbb{R}$ (see [24] for the case $n = 2$). Second, one can assume that only the points of the set \mathcal{K}_1 or the set $\mathcal{K} \cap \partial Q$ are singular points in the definition of weighted spaces, which allows one to study nonlocal elliptic problems only for $a > l + 2m - 1$ (see [26, 27]).

It is proved in [20, 22] that “local” elliptic problems in bounded domains have the Fredholm property (see Definition 2.1) if some model elliptic operators (depending on a parameter ω) in plane angles have a trivial kernel and cokernel for all $\omega \in S^{n-3}$, where

$$S^{n-3} = \{\omega \in \mathbb{R}^{n-2} : |\omega| = 1\}.$$

Similarly, elliptic operators in $\mathbb{R}^2 \setminus \{0\}$ with the parameter $\omega \in S^{n-3}$ arise if the points of the set $\mathcal{K} \cap Q$ are considered as singular points in the definition of weighted spaces. However, we prove in this paper that these operators are not isomorphisms, see Sec. 3. Therefore, unlike the case of plane domains, if $n \geq 3$ and $\mathcal{K} \cap Q \neq \emptyset$, only the points of the set \mathcal{K}_1 or the set $\mathcal{K} \cap \partial Q$ can be considered as singular points. This leads to the restriction $a > l + 2m - 1$ (see Sec. 4.1 for details).

The paper is organized as follows.

The setting of nonlocal elliptic problems is presented in Sec. 1. In the same section, we define model problems in dihedral angles and introduce function spaces. In Sec. 2, we consider the solvability of nonlocal problems in dihedral angles. In particular, we give an example of a nonlocal elliptic problem in a dihedral angle which is uniquely solvable on weighted spaces for $0 \leq a \leq 2$. In Sec. 3, we show that an elliptic operator of order $2m$ acting from $H_a^{l+2m}(\mathbb{R}^n)$ to $H_a^l(\mathbb{R}^n)$ is not an isomorphism for any $a \in \mathbb{R}$ and integer $l \geq 0$. Here we suppose that the points of the set

$$\mathcal{P} = \{x = (y, z) \in \mathbb{R}^n : y = 0, z \in \mathbb{R}^{n-2}\}$$

are singular in the definition of weighted spaces $H_a^{l+2m}(\mathbb{R}^n)$ and $H_a^l(\mathbb{R}^n)$. In Sec. 4, we prove a priori estimates for solutions of nonlocal problems in bounded domains. In Sec. 5, we construct a right regularizer for those problems. Thus, we prove a theorem on the Fredholm solvability of nonlocal elliptic problems in bounded domains. Section 6 is devoted to generalizations of nonlocal elliptic problems to the case of nonlocal terms supported on the manifolds $\omega_{is}(\Gamma_i)$ near the set \mathcal{K}_1 and abstract nonlocal terms supported outside the set \mathcal{K}_1 . In Secs. 7 and 8, we prove the unique solvability of nonlocal elliptic problems with a parameter.

Note that it was A. V. Bitzadze and A. A. Samarskii [3] who first considered an elliptic equation with nonlocal conditions imposed on the shifts of different parts of the boundary of rectangular. In the general situation, they formulated this problem as an unsolved problem. Solvability and regularity of solutions for higher-order elliptic equations with general nonlocal conditions supported near the boundary were studied in [24–27]. Second-order elliptic equations with nonlocal conditions near the boundary were also considered in [11, 12, 14]. One can find various applications of nonlocal elliptic problems as well as comprehensive bibliography of the question in [8, 24, 27, 28].

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1 Setting of Nonlocal Elliptic Problems

1.1

Let $Q \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded domain with boundary $\partial Q = \bigcup_{i=1}^{N_0} \overline{\Gamma}_i$, where Γ_i are open connected, in the topology of ∂Q , $(n-1)$ -dimensional C^∞ manifolds. Assume that, in a neighborhood of each point $g \in \overline{\Gamma}_i \setminus \Gamma_i$, the domain Q is diffeomorphic to an n -dimensional dihedral angle

$$\Theta = \{x = (y, z) \in \mathbb{R}^n : d_1 < \varphi < d_2, z \in \mathbb{R}^{n-2}\},$$

where r, φ are the polar coordinates of the point $y \in \mathbb{R}^2$, $d_j = d_j(g)$, $j = 1, 2$.

Introduce the differential operators

$$A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha, \quad B_{i\mu s}(x, D) = \sum_{|\alpha| \leq m_{i\mu}} b_{i\mu s \alpha}(x) D^\alpha,$$

where $a_\alpha, b_{i\mu s \alpha} \in C^\infty(\mathbb{R}^n)$ are complex-valued functions ($i = 1, \dots, N_0$; $\mu = 1, \dots, m$; $s = 0, \dots, S_i$), $m_{i\mu} \leq 2m - 1$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $D_j = -i\partial/\partial x_j$. If it is necessary to indicate the variables with respect to which a function u is differentiated, we write $D_y^\alpha u$, $D_z^\alpha u$, etc.

Let ω_{is} ($i = 1, \dots, N_0$; $s = 1, \dots, S_i$) denote a C^∞ diffeomorphism mapping some neighborhood Ω_i of the manifold Γ_i onto the set $\omega_{is}(\Omega_i)$ in such a way that $\omega_{is}(\Gamma_i) \subset Q$. Assume that the set

$$\mathcal{K} = \left\{ \bigcup_i (\overline{\Gamma}_i \setminus \Gamma_i) \right\} \cup \left\{ \bigcup_{i,s} \omega_{is}(\overline{\Gamma}_i \setminus \Gamma_i) \right\} \cup \left\{ \bigcup_{j,p} \bigcup_{i,s} \omega_{jp}(\omega_{is}(\overline{\Gamma}_i \setminus \Gamma_i) \cap \Gamma_j) \right\} \quad (1.1)$$

can be represented as follows:

$$\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3, \quad (1.2)$$

where $\mathcal{K}_1 = \bigcup_{\nu} \mathcal{K}_{1\nu} = \partial Q \setminus \bigcup_i \Gamma_i$, $\mathcal{K}_2 = \bigcup_{\nu} \mathcal{K}_{2\nu} \subset \bigcup_i \Gamma_i$, $\mathcal{K}_3 = \bigcup_{\nu} \mathcal{K}_{3\nu} \subset Q$ ($\nu = 1, \dots, N_j$; $j = 1, 2, 3$), $\mathcal{K}_{j\nu}$ are mutually disjoint $(n-2)$ -dimensional connected C^∞ manifolds without a boundary. In particular, the sets \mathcal{K}_2 and \mathcal{K}_3 may be empty.

We study the following nonlocal problem:

$$Au \equiv A(x, D)u(x) = f_0(x), \quad x \in Q, \quad (1.3)$$

$$B_{i\mu}u \equiv \sum_{s=0}^{S_i} (B_{i\mu s}(x, D)u)(\omega_{is}(x))|_{\Gamma_i} = f_{i\mu}(x), \quad x \in \Gamma_i; \quad i = 1, \dots, N_0; \quad \mu = 1, \dots, m, \quad (1.4)$$

where $(B_{i\mu s}(x, D)u)(\omega_{is}(x))|_{\Gamma_i} = B_{i\mu s}(\hat{x}, D)u(\hat{x})|_{\hat{x}=\omega_{is}(x)}$, $x \in \Gamma_i$, $\omega_{i0}(x) \equiv x$.

We assume throughout that the operators $A(x, D)$ and $B_{i\mu 0}(x, D)$ satisfy the following conditions.

Condition 1.1. *The operator $A(x, D)$ is properly elliptic for each $x \in \overline{Q}$.*

Condition 1.2. *The system $\{B_{i\mu 0}(x, D)\}_{\mu=1}^m$ satisfies the Lopatinskiĭ condition with respect to $A(x, D)$ and is normal for all $i = 1, \dots, N_0$ and $x \in \overline{\Gamma_i}$ (see [19, Chap. 2, Sec. 1.4]).*

Denote by ω_{is}^{+1} the transformation $\omega_{is} : \Omega_i \rightarrow \omega_{is}(\Omega_i)$ and by $\omega_{is}^{-1} : \omega_{is}(\Omega_i) \rightarrow \Omega_i$ the inverse transformation.

Definition 1.1. The set of all points that can be obtained from a point $g \in \mathcal{K}_1$ by consecutively applying to it the transformations ω_{is}^{+1} or ω_{is}^{-1} mapping the points of the set \mathcal{K}_1 to \mathcal{K}_1 is called an *orbit* of the point g and denoted by $\mathcal{O}(g)$.

In other words, the orbit $\mathcal{O}(g)$ consists of the point $g \in \mathcal{K}_1$ and the points that can be obtained from g in the following way: given a point $h \in \mathcal{O}(g)$, the point $\omega_{is}(h)$ belongs to $\mathcal{O}(g)$ iff $h \in \overline{\Gamma_i} \cap \mathcal{K}_1$ and $\omega_{is}(h) \in \mathcal{K}_1$, while the point $\omega_{is}^{-1}(h)$ belongs to $\mathcal{O}(g)$ iff $h \in \omega_{is}(\overline{\Gamma_i}) \cap \mathcal{K}_1$.

Assume that the following condition holds.

Condition 1.3 (finiteness of the orbits). *1. For each point $g \in \mathcal{K}_1$, the orbit $\mathcal{O}(g)$ consists of finitely many points g_j , $j = 1, \dots, N = N(g)$.*

2. There are neighborhoods $\hat{V}(g_j) \subset V(g_j) \subset \mathbb{R}^n \setminus (\mathcal{K}_2 \cup \mathcal{K}_3)$, $V(g_j) \cap V(g_p) = \emptyset$ ($j \neq p$), of the points $g_j \in \mathcal{O}(g)$ such that, if $g_j \in \overline{\Gamma_i}$ and $\omega_{is}(g_j) = g_p$, then $\hat{V}(g_j) \subset \Omega_i$ and $\omega_{is}(\hat{V}(g_j)) \subset V(g_p)$.

The following condition means that the support of nonlocal terms intersects the boundary at the points of the set \mathcal{K}_1 in a nontangential way.

Condition 1.4 (nontangential approach). *For each point $g \in \mathcal{K}_1$ and $j = 1, \dots, N(g)$, there exists a smooth nondegenerate change of variables $x \rightarrow x' = x'(g, j)$ such that the neighborhood $V(g_j)$ reduces, under this change of variables, to some neighborhood of the origin $V(0)$ and, moreover:*

- 1. The sets $Q \cap V(g_j)$ and $\Gamma_i \cap V(g_j)$ reduce to the intersection of a dihedral angle Θ_j with $V(0)$ and to the intersection of a side $\Gamma_{j\rho}$ ($\rho = 1$ or $\rho = 2$) of Θ_j with $V(0)$, respectively;*
- 2. Each transformation $\omega_{is}(x)$, for $x \in \hat{V}(g_j)$, $g_j \in \overline{\Gamma_i}$, reduces to the composition of rotation and homothety on the plane $\{y'\}$ in the new variables $x' = (y', z')$, where $y' \in \mathbb{R}^2$ and $z' \in \mathbb{R}^{n-2}$.*

Remark 1.1. If $N(g) = 1$, then the orbit of $g \in \mathcal{K}_1$ consists of the unique point g . This is the case iff the following two conditions are fulfilled:

- 1. $\omega_{is}(g) = g$ for all i and s such that $g \in \overline{\Gamma_i} \cap \mathcal{K}_1$ and $\omega_{is}(g) \in \mathcal{K}_1$;*
- 2. there do not exist indices i and s and a point $h \in \overline{\Gamma_i} \cap \mathcal{K}_1$ such that $h \neq g$ and $\omega_{is}(h) = g$.*

1.2

For any domain Ω , denote by $W^k(\Omega) = W_2^k(\Omega)$ ($k \geq 0$ is an integer) the Sobolev space. Denote by $W^{k-1/2}(\Gamma)$ ($k \geq 1$ is an integer) the space of traces on a smooth $(n-1)$ -dimensional manifold $\Gamma \subset \overline{\Omega}$, with the norm

$$\|\psi\|_{W^{k-1/2}(\Gamma)} = \inf \|v\|_{W^k(\Omega)} \quad (v \in W^k(\Omega) : v|_{\Gamma} = \psi).$$

If X is a domain in \mathbb{R}^n , $n = 1, 2, \dots$, we denote by $C_0^\infty(X)$ the set of functions infinitely differentiable on \overline{X} and compactly supported on X . If M is a union of finitely many $(n-l)$ -dimensional manifolds ($l = 1, \dots, n$) lying in \overline{X} , we denote by $C_0^\infty(\overline{X} \setminus M)$ the set of functions infinitely differentiable on \overline{X} and compactly supported on $\overline{X} \setminus M$.

Now we introduce different weighted spaces for different domains Ω . Consider the following cases:

1. $\Omega = Q$; denote either $K = \mathcal{K}_1$ or $K = \mathcal{K}_1 \cup \mathcal{K}_2$ (cf. Condition 4.1 in Sec. 4), and let $\rho(x)$ be a function such that $\rho \in C^\infty(\mathbb{R}^n \setminus K)$, $\rho(x) > 0$ for $x \in \mathbb{R}^n \setminus K$ and it is equivalent, in a neighborhood of the set K , to the distance from a point $x \in \Omega$ to the set K ;
2. $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega \in C^\infty$; denote by K some $(n-2)$ -dimensional manifold of class C^∞ lying in $\bar{\Omega}$, and let ρ be the same function as in case 1;
3. Ω is an n -dimensional dihedral angle Θ ; denote $K = \mathcal{P}$, where

$$\mathcal{P} = \{x = (y, z) \in \mathbb{R}^n : y = 0, z \in \mathbb{R}^{n-2}\},$$

and let $\rho(x) = |y|$;

4. $\Omega = \mathbb{R}^n$; denote $K = \mathcal{P}$, and let $\rho(x) = |y|$.

Introduce the weighted space $H_a^k(\Omega) = H_a^k(\Omega, K)$ as the completion of the set $C_0^\infty(\bar{\Omega} \setminus K)$ with respect to the norm

$$\|u\|_{H_a^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_{\Omega} \rho^{2(a-k+|\alpha|)} |D^\alpha u|^2 dx \right)^{1/2}, \quad (1.5)$$

where $k \geq 0$ is an integer and $a \in \mathbb{R}$.

Denote by $H_a^{k-1/2}(\Gamma)$ ($k \geq 1$ is an integer) the space of traces on a smooth $(n-1)$ -dimensional manifold $\Gamma \subset \bar{\Omega}$, with the norm

$$\|\psi\|_{H_a^{k-1/2}(\Gamma)} = \inf \|v\|_{H_a^k(\Omega)} \quad (v \in H_a^k(\Omega) : v|_\Gamma = \psi). \quad (1.6)$$

One can similarly introduce the weighted spaces $H_a^k(\Omega)$ and $H_a^{k-1/2}(\Gamma)$ for $n = 2$. In particular, we set $K = \{0\}$ for $\Omega = \theta = \{y \in \mathbb{R}^2 : d_1 < \varphi < d_2, 0 < r\}$ or $\Omega = \mathbb{R}^2$.

In what follows, we assume that $u \in H_a^{l+2m}(Q)$ and $f = \{f_0, f_{i\mu}\} \in \mathcal{H}_a^l(Q, \Gamma)$ in problem (1.3), (1.4), where

$$\mathcal{H}_a^l(Q, \Gamma) = H_a^l(Q) \times \prod_{i, \mu} H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)$$

and $l \geq 0$ is an integer.

1.3

Fix an arbitrary point $g \in \mathcal{K}_1$. By Condition 1.3, the orbit $\mathcal{O}(g)$ consists of finitely many points g_j , $j = 1, \dots, N = N(g)$. We now reduce problem (1.3), (1.4) to a system of N elliptic equations in dihedral angles with nonlocal boundary-value conditions. To do this, we suppose that

$$\text{supp } u \subset \left(\bigcup_j \hat{V}(g_j) \right) \cap \bar{Q}.$$

Denote by $u_j(x)$ the function $u(x)$ for $x \in Q \cap V(g_j)$. If $g_j \in \bar{\Gamma}_i$ and $x \in \hat{V}(g_j)$, then $\omega_{is}(x) \in V(g_p)$ for some p , $1 \leq p \leq N$, by Condition 1.3. Denote the function $u(\omega_{is}(x))$ by $u_p(\omega_{is}(x))$. It is clear that $u(\omega_{i0}(x)) = u(x) = u_j(x)$. In the above notation, problem (1.3), (1.4) takes the form

$$A(x, D)u_j(x) = f_0(x), \quad x \in Q \cap \hat{V}(g_j), \quad (1.7)$$

$$\sum_{s \in S_{ij}^g} (B_{i\mu s}(x, D)u_p)(\omega_{is}(x))|_{\Gamma_i} = f_{i\mu}(x), \quad x \in \hat{V}(g_j) \cap \Gamma_i; \quad (1.8)$$

$$i \in \{1 \leq i \leq N_0 : \hat{V}(g_j) \cap \Gamma_i \neq \emptyset\}; \quad j = 1, \dots, N; \quad \mu = 1, \dots, m,$$

where $S_{ij}^g = \{0 \leq s \leq S_i : \omega_{is}(g_j) = g_p \in \mathcal{O}(g) \text{ for some } p = 1, \dots, N\}$.

Using the change of variables $x \rightarrow x'(g, j)$ from Sec. 1.1, we introduce the functions $v_j(x') = u_j(x(x'))$. By Condition 1.4, problem (1.7), (1.8) takes the following form:

$$A_j(x', D_{y'}, D_{z'})v_j(x') = f_j(x'), \quad x' \in \Theta_j; \quad j = 1, \dots, N, \quad (1.9)$$

$$\sum_{k=1}^N \sum_{s=0}^{S_{j\rho k}} (B_{j\rho\mu ks}(x', D_{y'}, D_{z'})v_k)(\mathcal{G}_{j\rho ks}y', z')|_{\Gamma_{j\rho}} = f_{j\rho\mu}(x'), \quad x' \in \Gamma_{j\rho}; \quad (1.10)$$

$$j = 1, \dots, N; \quad \rho = 1, 2; \quad \mu = 1, \dots, m.$$

Here the operators A_j and $B_{j\rho\mu ks}$ have variable coefficients of class C^∞ ;

$$\Theta_j = \{x' = (y', z') : 0 < d_{j1} < \varphi < d_{j2}, z' \in \mathbb{R}^{n-2}\},$$

$$\Gamma_{j\rho} = \{x' = (y', z') : \varphi = d_{j\rho}, z' \in \mathbb{R}^{n-2}\};$$

$\mathcal{G}_{j\rho ks}$ is the operator of rotation by an angle $\varphi_{j\rho ks}$ and homothety with a coefficient $\chi_{j\rho ks}$ in the y' -plane so that $d_{k1} < d_{j\rho} + \varphi_{j\rho ks} < d_{k2}$ and $0 < \chi_{j\rho ks}$ for $(k, s) \neq (j, 0)$, while $\varphi_{j\rho j0} = 0$ and $\chi_{j\rho j0} = 1$ (i.e., $\mathcal{G}_{j\rho j0}y' \equiv y'$); $v = (v_1, \dots, v_N)$.

Remark 1.2. If $g \in \overline{\Gamma_i}$, $N = N(g) = 1$ (cf. Remark 1.1), and $\omega_{is}(g) \neq g$ for all $s = 1, \dots, S_i$, then model problem (1.9), (1.10) contains no nonlocal terms due to the fact that the manifolds $\mathcal{K}_{j\nu}$ are mutually disjoint.

Introduce the following spaces of vector-valued functions:

$$\mathcal{H}_a^k(\Theta) = \prod_{j=1}^N H_a^k(\Theta_j), \quad \mathcal{H}_a^l(\Theta, \Gamma) = \mathcal{H}_a^l(\Theta) \times \prod_{j=1}^N \prod_{\rho=1,2} \prod_{\mu=1}^m H_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho}),$$

where $m_{j\rho\mu}$ is the order of the operator $B_{j\rho\mu ks}(x', D_{y'}, D_{z'})$.

Consider the linear bounded operator $\mathcal{L}_g : \mathcal{H}_a^{l+2m}(\Theta) \rightarrow \mathcal{H}_a^l(\Theta, \Gamma)$ given by

$$\mathcal{L}_g v = \left\{ A_j(D_{y'}, D_{z'}) v_j(y', z'), \sum_{k=1}^N \sum_{s=0}^{S_{j\rho k}} (B_{j\rho\mu ks}(D_{y'}, D_{z'}) v_k)(\mathcal{G}_{j\rho ks} y', z')|_{\Gamma_{j\rho}} \right\}, \quad (1.11)$$

where $A_j(D_{y'}, D_{z'})$ and $B_{j\rho\mu ks}(D_{y'}, D_{z'})$ are principal homogeneous parts of the operators $A_j(0, D_{y'}, D_{z'})$ and $B_{j\rho\mu ks}(0, D_{y'}, D_{z'})$, respectively. The subscript g means that the operator \mathcal{L}_g depends on the choice of the point $g \in \mathcal{K}_1$ (and therefore, it depends on the orbit $\mathcal{O}(g)$). Clearly, each of the operators $A_j(D_{y'}, D_{z'})$ is properly elliptic, while the system $\{B_{j\rho\mu j0}(D_{y'}, D_{z'})\}_{\mu=1}^m$ satisfies the Lopatinski condition with respect to $A_j(D_{y'}, D_{z'})$ and is normal for all $j = 1, \dots, N$ and $\rho = 1, 2$.

Example 1.1. Let $Q \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial Q \in C^\infty$ which is a surface of revolution about the axis x_3 . Set $P = \{(0, 0, 3)\} \cup \{(0, 0, -3)\} \cup \{x : x_3 = 0, \sqrt{x_1^2 + x_2^2} = 3\}$ and $P^{1/4} = \{x : \text{dist}(x, P) < 1/4\}$. Assume that, outside the set $P^{1/4}$, the boundary ∂Q coincides with the boundary of the domain

$$\left\{ x : x_3 < 3 - \sqrt{x_1^2 + x_2^2} \right\} \cap \left\{ x : x_3 > -3 + \sqrt{x_1^2 + x_2^2} \right\}.$$

Denote

$$\Gamma_1 = \{x \in \partial Q : x_3 < -2\}, \quad \Gamma_2 = \{x \in \partial Q : x_3 > 2\}, \quad \Gamma_3 = \partial Q \setminus (\overline{\Gamma_1} \cup \overline{\Gamma_2}).$$

We consider the following nonlocal boundary-value problem:

$$-\Delta u = f_0(x), \quad x \in Q, \quad (1.12)$$

$$\begin{aligned} [u(x) + \alpha_j u(x + h_j) + \beta_j u(\mathcal{G}_\pi x + h_j)]|_{\Gamma_j} &= 0, \quad j = 1, 2, \\ u(x)|_{\Gamma_3} &= 0, \end{aligned} \quad (1.13)$$

where $\alpha_j, \beta_j \in \mathbb{R}$, $h_j = (-1)^{j+1}(0, 0, 4)$, $j = 1, 2$, and \mathcal{G}_π is the operator of rotation by the angle π about the axis x_3 . Clearly, we have (see Fig. 1.1)

$$\mathcal{K} = \mathcal{K}_1 = \mathcal{K}_{11} \cup \mathcal{K}_{12}, \quad \mathcal{K}_{1\nu} = \{x \in \partial Q : x_3 = (-1)^\nu 2\}, \quad \nu = 1, 2.$$

The orbit of each point $g \in \mathcal{K}_{11}$ consists of the four points: $g_1 = g$, $g_2 = \mathcal{G}_\pi g_1$, $g_3 = g_1 + h_1$, and $g_4 = \mathcal{G}_\pi g_1 + h_1$. Let $\hat{V}(g_j) = V(g_j) = \{x : |x - g_j| < \varepsilon\}$, where ε is sufficiently small, and let $\text{supp } u \subset \left(\bigcup_j V(g_j)\right) \cap \overline{Q}$. For $x \in V(g_j)$, we introduce the new variables $x' = (y'_1, y'_2, z')$ by the formulas

$$y'_1 = r - 1, \quad y'_2 = x_3 - g_{j3}, \quad z' = \varphi - \varphi_j,$$

where r, φ, x_3 and $1, \varphi_j, g_{j3}$ are the cylindrical coordinates of the points x and g_j , respectively. Clearly, the transformation $x \mapsto x'(g, j)$ is nondegenerate for $x \neq 0$ and each open set $V(g_j)$ is taken onto one and the same neighborhood of the origin $V(0)$ under this transformation. We define the vector-valued function $v(x')$ such that $v_j(x') = u_j(x(x'))$ for $x' \in V(0)$, where $u_j(x) = u(x)$ for $x \in V(g_j) \cap Q$. Denote $x' = (y'_1, y'_2, z')$ by $x = (y_1, y_2, z)$ again. Then the boundary-value problem (1.12), (1.13) takes the form (see Fig. 1.2)

$$-\frac{\partial^2 v_j}{\partial y_1^2} - \frac{\partial^2 v_j}{\partial y_2^2} - \frac{1}{(1 + y_1)^2} \frac{\partial^2 v_j}{\partial z^2} - \frac{1}{1 + y_1} \frac{\partial v_j}{\partial y_1} = f_j(x), \quad x \in \Theta_j, \quad j = 1, \dots, 4, \quad (1.14)$$

$$\begin{aligned} v_j|_{\Gamma_{j1}} &= 0, \quad j = 1, \dots, 4, \\ (v_1 + \alpha_1 v_3 + \beta_1 v_4)|_{\Gamma_{12}} &= (v_2 + \beta_1 v_3 + \alpha_1 v_4)|_{\Gamma_{22}} = 0, \\ (v_3 + \alpha_2 v_1 + \beta_2 v_2)|_{\Gamma_{32}} &= (v_4 + \beta_2 v_1 + \alpha_2 v_2)|_{\Gamma_{42}} = 0. \end{aligned} \quad (1.15)$$

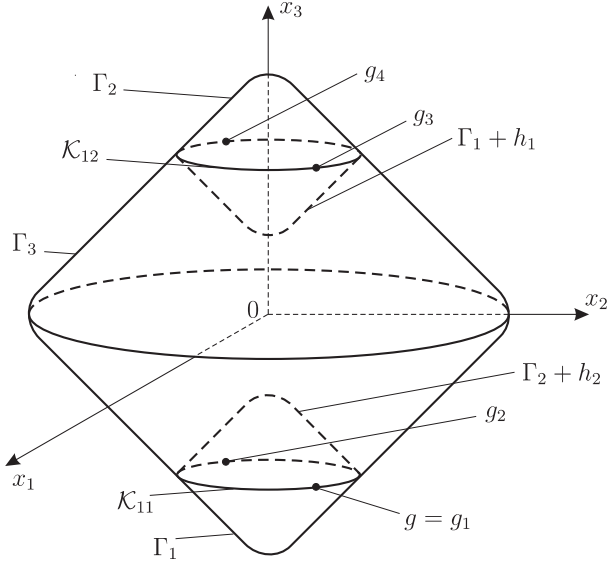


Figure 1.1: Problem (1.12), (1.13)

Here

$$\begin{aligned}
 \Theta_1 = \Theta_2 &= \{x \in \mathbb{R}^3 : y_2 > y_1\}, & \Theta_3 = \Theta_4 &= \{x \in \mathbb{R}^3 : y_2 < -y_1\}, \\
 \Gamma_{11} = \Gamma_{21} &= \{x \in \mathbb{R}^3 : y_2 = y_1, y_1 > 0\}, & \Gamma_{31} = \Gamma_{41} &= \{x \in \mathbb{R}^3 : y_2 = -y_1, y_1 > 0\}, \\
 \Gamma_{12} = \Gamma_{22} &= \{x \in \mathbb{R}^3 : y_2 = y_1, y_1 < 0\}, & \Gamma_{32} = \Gamma_{42} &= \{x \in \mathbb{R}^3 : y_2 = -y_1, y_1 < 0\}.
 \end{aligned}$$

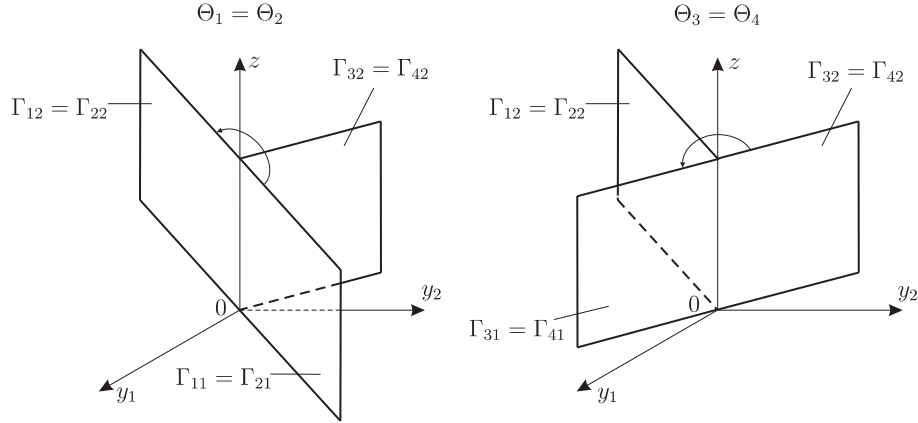


Figure 1.2: Problem (1.14), (1.15)

Clearly, Conditions 1.1–1.4 hold in this example.

Passing to the principal homogeneous parts in Eqs. (1.14) and freezing the coefficients at the origin, we obtain

$$-\Delta v_j = f_j(x), \quad x \in \Theta_j, \quad j = 1, \dots, 4.$$

Nonlocal boundary conditions (1.15) do not change.

1.4

Now fix an arbitrary point $g \in \mathcal{K}_2$. Clearly, $g \in \mathcal{K}_{2\nu} \cap \Gamma_i$ for some $1 \leq \nu \leq N_2$ and $1 \leq i \leq N_0$. By virtue of the smoothness of Γ_i and $\mathcal{K}_{2\nu}$, there exists a C^∞ diffeomorphism $x \rightarrow x' = x'(g)$ defined on a small neighborhood $V(g)$ of the point g , such that the images of $Q \cap V(g)$ and $\mathcal{K}_{2\nu} \cap V(g)$ are the intersection of the half-space $\mathbb{R}_+^n = \{x : |\varphi| < \pi/2, z \in \mathbb{R}^{n-2}\}$ with some neighborhood $V(0)$ and the intersection of the set \mathcal{P} with $V(0)$, respectively.

Let $A(D_{y'}, D_{z'})$ and $B_{i\mu 0}(D_{y'}, D_{z'})$ be the principal homogeneous parts of the operators $A(g, D_y, D_z)$ and $B_{i\mu 0}(g, D_y, D_z)$, respectively, written in the new coordinates $x' = x'(g)$.

We introduce the linear bounded operator

$$\begin{aligned}\mathcal{L}_g &: H_a^{l+2m}(\mathbb{R}_+^n) \rightarrow \mathcal{H}_a^l(\mathbb{R}_+^n, \Gamma) \\ &= H_a^l(\mathbb{R}_+^n) \times H_a^{l+2m-m_{i\mu}-1/2}(\mathbb{R}_-^{n-1}) \times H_a^{l+2m-m_{i\mu}-1/2}(\mathbb{R}_+^{n-1})\end{aligned}$$

given by

$$\begin{aligned}\mathcal{L}_g u &= (A(D_{y'}, D_{z'})u(y', z'), \\ &\quad B_{i\mu 0}(D_{y'}, D_{z'})u(y', z')|_{\varphi=-\pi/2}, B_{i\mu 0}(D_{y'}, D_{z'})u(y', z')|_{\varphi=\pi/2}),\end{aligned}\tag{1.16}$$

where $\mathbb{R}_\pm^{n-1} = \{x' = (y', z') \in \mathbb{R}^n : \varphi = \pm\pi/2, z' \in \mathbb{R}^{n-2}\}$.

2 Nonlocal Elliptic Problems in Dihedral Angles

2.1

When studying nonlocal problems in bounded domains, we will represent the nonlocal operators as the sum of three operators. The first operator will correspond to nonlocal terms supported near the set \mathcal{K}_1 , the second operator to nonlocal terms supported outside the set \mathcal{K}_1 , and the third one to lower-order terms (compact perturbations). In this section, we consider a model operator corresponding to the problem with nonlocal terms supported near the set \mathcal{K}_1 .

By using the Fourier transform with respect to z , one can reduce the study of the operator \mathcal{L}_g in dihedral angles to the study of a model operator $\mathcal{L}_g(\omega)$ in plane angles, where ω is a parameter belonging to the unit sphere

$$S^{n-3} = \{\omega \in \mathbb{R}^2 : |\omega| = 1\},$$

see [5, 25]. In this section, we formulate some results (mostly proved in [5, 25]) which we need below and illustrate them by an example. Note that the Fourier transform approach was earlier proposed for the study of local elliptic problems in dihedral angles [20].

To introduce the operator $\mathcal{L}_g(\omega)$, we preliminarily consider weighted spaces with nonhomogeneous weight. Denote by $E_a^k(\Omega)$ the completion of the set $C_0^\infty(\overline{\Omega} \setminus \{0\})$ with respect to the norm

$$\|u\|_{E_a^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_{\Omega} r^{2a} (r^{2(|\alpha|-k)} + 1) |D_y^\alpha u(y)|^2 dy \right)^{1/2},$$

where either $\Omega = \theta = \{y \in \mathbb{R}^2 : d_1 < \varphi < d_2\}$ or $\Omega = \mathbb{R}^2$; r, φ are the polar coordinates of the point y ; $k \geq 0$ is an integer. Let $\gamma \subset \overline{\Omega}$ be a half-line given by $\gamma = \{y \in \mathbb{R}^2 : \varphi = \varphi_0\}$, where $d_1 \leq \varphi_0 \leq d_2$ for $\Omega = \theta$. Denote by $E_a^{k-1/2}(\gamma)$ ($k \geq 1$ is an integer) the space of traces on γ with the norm

$$\|\psi\|_{E_a^{k-1/2}(\gamma)} = \inf \|v\|_{E_a^k(\Omega)} \quad (v \in E_a^k(\Omega) : v|_\gamma = \psi).$$

Introduce the following spaces of vector-valued functions:

$$\mathcal{E}_a^k(\theta) = \prod_{j=1}^N E_a^k(\theta_j), \quad \mathcal{E}_a^l(\theta, \gamma) = \mathcal{E}_a^l(\theta) \times \prod_{j=1}^N \prod_{\rho=1,2} \prod_{\mu=1}^m E_a^{l+2m-m_{j\rho\mu}-1/2}(\gamma_{j\rho}),$$

where $\theta_j = \{y \in \mathbb{R}^2 : d_{j1} < \varphi < d_{j2}\}$ and $\gamma_{j\rho} = \{y \in \mathbb{R}^2 : \varphi = d_{j\rho}\}$.

For a fixed point $g \in \mathcal{K}_1$, we consider the linear bounded operator

$$\mathcal{L}_g(\omega) : \mathcal{E}_a^{l+2m}(\theta) \rightarrow \mathcal{E}_a^l(\theta, \gamma)$$

given by

$$\mathcal{L}_g(\omega)V = \left\{ A_j(D_y, \omega)V_j(y), \sum_{k,s} (B_{j\rho\mu ks}(D_y, \omega)V_k)(\mathcal{G}_{j\rho ks}y)|_{\gamma_{j\rho}} \right\}, \tag{2.1}$$

where $\omega \in S^{n-3}$ and $V = (V_1, \dots, V_N)$, cf. (1.11).

2.2

We write the operators $A_j(D_y, 0)$ and $B_{j\rho\mu ks}(D_y, 0)$ in the polar coordinates:

$$A_j(D_y, 0) = r^{-2m} \hat{A}_j(\varphi, D_\varphi, rD_r), \quad B_{j\rho\mu ks}(D_y, 0) = r^{-m_{j\rho\mu}} \hat{B}_{j\rho\mu ks}(\varphi, D_\varphi, rD_r),$$

where $D_\varphi = -i\partial/\partial\varphi$, $D_r = -i\partial/\partial r$.

Introduce the following spaces of vector-valued functions:

$$\mathcal{W}^k(d_1, d_2) = \prod_{j=1}^N \mathcal{W}^k(d_{j1}, d_{j2}), \quad \mathcal{W}^l[d_1, d_2] = \mathcal{W}^l(d_1, d_2) \times \mathbb{C}^{mN} \times \mathbb{C}^{mN}.$$

Consider the analytic operator-valued function $\hat{\mathcal{L}}_g(\lambda) : \mathcal{W}^{l+2m}(d_1, d_2) \rightarrow \mathcal{W}^l[d_1, d_2]$ given by

$$\begin{aligned} \hat{\mathcal{L}}_g(\lambda)w = & \left\{ \hat{A}_j(\varphi, D_\varphi, \lambda)w_j, \right. \\ & \left. \sum_{k,s} e^{(i\lambda - m_{j\rho\mu}) \ln \chi_{j\rho ks}} (\hat{B}_{j\rho\mu ks}(\varphi, D_\varphi, \lambda)w_k)(\varphi + \varphi_{j\rho ks})|_{\varphi=d_{j\rho}} \right\}, \end{aligned} \quad (2.2)$$

where $w = (w_1, \dots, w_N)$.

By Lemmas 2.1 and 2.2 in [25], there exists a finite-meromorphic operator-valued function $\hat{\mathcal{R}}_g(\lambda) : \mathcal{W}^l[d_1, d_2] \rightarrow \mathcal{W}^{l+2m}(d_1, d_2)$ such that $\hat{\mathcal{L}}_g^{-1}(\lambda) = \hat{\mathcal{R}}_g(\lambda)$ for any λ which is not a pole of $\hat{\mathcal{R}}_g(\lambda)$. Moreover, if $\lambda_0 = \mu_0 + i\nu_0$ is a pole of $\hat{\mathcal{R}}_g(\lambda)$, then λ_0 is an eigenvalue of $\hat{\mathcal{L}}_g(\lambda)$, and there exists a number $\delta > 0$ such that the set $\{\lambda \in \mathbb{C} : 0 < |\operatorname{Im} \lambda - \nu_0| < \delta\}$ contains no eigenvalues of $\hat{\mathcal{L}}_g(\lambda)$.

2.3

Definition 2.1. Let H_1 and H_2 denote Hilbert spaces. A linear bounded operator $L : H_1 \rightarrow H_2$ is said to have the *Fredholm property* if $\dim \mathcal{N}(L) < \infty$, $\operatorname{codim} \mathcal{R}(L) < \infty$, and $\mathcal{R}(L)$ is closed, where $\mathcal{N}(L)$ and $\mathcal{R}(L)$ are the kernel and the image of the operator L , respectively.

The following theorem shows that spectral properties of the operator-valued function $\hat{\mathcal{L}}_g(\lambda)$ affect whether or not the operator $\mathcal{L}_g(\omega)$ has the Fredholm property.

Theorem 2.1. *Let Conditions 1.1–1.4 hold. If the line $\operatorname{Im} \lambda = a + 1 - l - 2m$ contains no eigenvalues of the operator-valued function $\hat{\mathcal{L}}_g(\lambda)$, then the operator $\mathcal{L}_g(\omega) : \mathcal{E}_a^{l+2m}(\theta) \rightarrow \mathcal{E}_a^l(\theta, \gamma)$ has the Fredholm property for all $\omega \in S^{n-3}$.*

If the operator $\mathcal{L}_g(\omega)$ has the Fredholm property for a certain $\omega \in S^{n-3}$, then the line $\operatorname{Im} \lambda = a + 1 - l - 2m$ contains no eigenvalues of $\hat{\mathcal{L}}_g(\lambda)$.

Theorem 2.1 was proved in [5]. This result is a generalization of Theorem 3.2 in [25], where one additionally assumes that the line $\operatorname{Im} \lambda = a + 1 - l - 2m$ contains no eigenvalues of the corresponding localized operator with a parameter λ .

The following theorem results from Theorems 3.3, 9.2, and 9.3 in [5].

Theorem 2.2. *Let Conditions 1.1–1.4 hold. Then the operator $\mathcal{L}_g : \mathcal{H}_a^{l+2m}(\Theta) \rightarrow \mathcal{H}_a^l(\Theta, \Gamma)$ is an isomorphism iff the operator $\mathcal{L}_g(\omega) : \mathcal{E}_a^{l+2m}(\theta) \rightarrow \mathcal{E}_a^l(\theta, \gamma)$ is an isomorphism for each $\omega \in S^{n-3}$.*

Denote

$$\mathcal{L}'_g v = \left\{ A_j^0(x, D_y, D_z)v_j(y, z), \sum_{k,s} (B_{j\rho\mu ks}^0(x, D_y, D_z)v_k)(\mathcal{G}_{j\rho ks}y, z)|_{\Gamma_{j\rho}} \right\}, \quad (2.3)$$

where $A_j^0(x, D_y, D_z)$ and $B_{j\rho\mu ks}^0(x, D_y, D_z)$ are principal homogeneous parts of the operators $A_j(x, D_y, D_z)$ and $B_{j\rho\mu ks}(x, D_y, D_z)$, respectively. Note that $A_j^0(0, D_y, D_z) = A_j(D_y, D_z)$ and $B_{j\rho\mu ks}^0(0, D_y, D_z) = B_{j\rho\mu ks}(D_y, D_z)$.

Let

$$B_\varepsilon = \{x \in \mathbb{R}^n : |x| < \varepsilon\}, \quad \varepsilon > 0,$$

be a ball of radius ε centered at the origin.

For each $\delta > 0$, we define a function $\eta = \eta_\delta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ for $x \in B_\delta$, $\operatorname{supp} \eta \subset B_{2\delta}$, and

$$|D^\beta \eta(x)| \leq k_1 \delta^{-|\beta|}, \quad x \in \mathbb{R}^n, \quad (2.4)$$

where $k_1 = k_1(\beta) > 0$ does not depend on δ .

The number δ will be specified in Secs. 4 and 5, where we prove a priori estimates and construct a right regularizer for the nonlocal problem in a bounded domain.

Introduce the linear bounded operator $\mathcal{L}''_g : \mathcal{H}_a^{l+2m}(\Theta) \rightarrow \mathcal{H}_a^l(\Theta, \Gamma)$ by the formula

$$\mathcal{L}''_g v = \mathcal{L}_g v + \eta(\mathcal{L}'_g - \mathcal{L}_g)v.$$

Corollary 2.1. *Let Conditions 1.1–1.4 hold. Assume that the line $\operatorname{Im} \lambda = a + 1 - l - 2m$ contains no eigenvalues of $\hat{\mathcal{L}}_g(\lambda)$ and $\dim \mathcal{N}(\mathcal{L}_g(\omega)) = \operatorname{codim} \mathcal{R}(\mathcal{L}_g(\omega)) = 0$ for any $\omega \in S^{n-3}$. Then the operator $\mathcal{L}''_g : \mathcal{H}_a^{l+2m}(\Theta) \rightarrow \mathcal{H}_a^l(\Theta, \Gamma)$ is an isomorphism for all sufficiently small $\delta > 0$ and $\|(\mathcal{L}''_g)^{-1}\| \leq c_0$, where $c_0 > 0$ does not depend on δ .*

Proof. Let us show that

$$\|\eta(\mathcal{L}'_g - \mathcal{L}_g)\| \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0. \quad (2.5)$$

To do so, we first prove that

$$\|\eta_1(B_{j\rho\mu ks}^0(x, D_y, D_z)u - B_{j\rho\mu ks}(D_y, D_z)u)\|_{H_a^{l+2m-m_{j\rho\mu}}(\Theta_k)} \leq k_2\delta\|u\|_{H_a^{l+2m}(\Theta_k)} \quad (2.6)$$

for all $u \in H_a^{l+2m}(\Theta_k)$, where $\eta_1(x) = \eta(\mathcal{G}_{j\rho ks}^{-1}y, z)$, while $k_2 > 0$ does not depend on u and δ .

Let

$$b\eta_1 D^\beta u, \quad |\beta| = m_{j\rho\mu},$$

be an arbitrary term of the expression

$$\eta_1(B_{j\rho\mu ks}^0(x, D_y, D_z)u - B_{j\rho\mu ks}(D_y, D_z)u),$$

where $b \in C^\infty(\mathbb{R}^n)$ and $b(0) = 0$. It follows from (2.4) and from the relation $b(0) = 0$ that

$$|r^{|\alpha|} D^\alpha(b\eta_1)| \leq k_3\delta, \quad x \in \mathbb{R}^n, \quad |\alpha| \leq l + 2m - m_{j\rho\mu}, \quad (2.7)$$

where $k_3 = k_3(\alpha) > 0$ does not depend on δ . Using (2.7) and the definition of the weighted spaces, we directly derive (2.6).

Analogous relations for the pairs of the operators $A_j^0(x, D_y, D_z)$ and $A_j(D_y, D_z)$ can be proved in the same way. Thus, we have proved (2.5).

It follows from the conditions of this corollary and from Theorem 2.1 that the operator $\mathcal{L}_g(\omega)$ is an isomorphism for any $\omega \in S^{n-3}$. Therefore, by Theorem 2.2, the operator \mathcal{L}_g is an isomorphism. Combining this fact with relation (2.5), we complete the proof. \square

2.4

In this subsection, we give an example of an operator which corresponds to a nonlocal elliptic problem in a dihedral angle and is an isomorphism.

Example 2.1. Let

$$\Theta = \{x = (y, z) \in \mathbb{R}^3 : 0 < \varphi < d, 0 < r, z \in \mathbb{R}\}$$

be a three-dimensional dihedral angle, where r, φ are the polar coordinates of the point y . Let

$$\Gamma_1 = \{x = (y, z) \in \mathbb{R}^3 : \varphi = 0, 0 < r, z \in \mathbb{R}\},$$

$$\Gamma_2 = \{x = (y, z) \in \mathbb{R}^3 : \varphi = d, 0 < r, z \in \mathbb{R}\}$$

be the sides of the angle Θ . Consider the nonlocal elliptic problem

$$-\Delta v(x) = f_0(x), \quad x \in \Theta, \quad (2.8)$$

$$\begin{aligned} v(\varphi, r, z)|_{\Gamma_1} - \alpha_1 v(\varphi + d/2, r, z)|_{\Gamma_1} &= f_1(x), & x \in \Gamma_1, \\ v(\varphi, r, z)|_{\Gamma_2} - \alpha_2 v(\varphi - d/2, r, z)|_{\Gamma_2} &= f_2(x), & x \in \Gamma_2, \end{aligned} \quad (2.9)$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$. Thus, the values of the unknown function v on the sides Γ_1 and Γ_2 are connected with the values of v on the half-plane $\{x = (y, z) \in \mathbb{R}^3 : \varphi = d/2, 0 < r, z \in \mathbb{R}\}$ lying strictly inside the angle Θ . The nonlocal transformations are rotations in y -plane only, while transformations with respect to the variables r and z are absent.

Introduce the linear bounded operator

$$\mathcal{L} : H_a^2(\Theta) \rightarrow \mathcal{H}_a^0(\Theta, \Gamma) = H_a^0(\Theta) \times H_a^{3/2}(\Gamma_1) \times H_a^{3/2}(\Gamma_2)$$

by the formula

$$\mathcal{L}v = (-\Delta v, v(\varphi, r, z)|_{\Gamma_1} - \alpha_1 v(\varphi + d/2, r, z)|_{\Gamma_1}, v(\varphi, r, z)|_{\Gamma_2} - \alpha_2 v(\varphi - d/2, r, z)|_{\Gamma_2}),$$

cf. (1.11). Along with the operator \mathcal{L} , we consider the linear bounded operator

$$\mathcal{L}(\omega) : E_a^2(\theta) \rightarrow \mathcal{E}_a^0(\theta, \gamma) = E_a^0(\theta) \times E_a^{3/2}(\gamma_1) \times E_a^{3/2}(\gamma_2)$$

given by

$$\mathcal{L}(\omega)V = (-\Delta_y V + V, V(\varphi, r)|_{\gamma_1} - \alpha_1 V(\varphi + d/2, r)|_{\gamma_1}, V(\varphi, r)|_{\gamma_2} - \alpha_2 V(\varphi - d/2, r)|_{\gamma_2}),$$

where

$$\theta = \{y \in \mathbb{R}^2 : 0 < \varphi < d, 0 < r\},$$

$$\gamma_1 = \{y \in \mathbb{R}^2 : \varphi = 0, 0 < r\}, \quad \gamma_2 = \{y \in \mathbb{R}^2 : \varphi = d, 0 < r\},$$

$\omega = \pm 1$, cf. (2.1). (Actually, one must write $-\Delta_y V + \omega^2 V$ instead of $-\Delta_y V + V$ in the definition of the operator $\mathcal{L}(\omega)$, but $\omega^2 = 1$ for $\omega = \pm 1$. Thus, the operator $\mathcal{L}(\omega)$ does not depend on ω in this example.)

The operator-valued function $\hat{\mathcal{L}}(\lambda) : W^2(0, d) \rightarrow \mathcal{W}^0[0, d] = L_2(0, d) \times \mathbb{C}^2$ corresponding to the operator $\mathcal{L}(\omega)$ has the form

$$\hat{\mathcal{L}}(\lambda)u = (-u_{\varphi\varphi} + \lambda^2 u, u|_{\varphi=0} - \alpha_1 u|_{\varphi=d/2}, u|_{\varphi=d} - \alpha_2 u|_{\varphi=d/2}),$$

cf. (2.2).

We prove that the operator $\mathcal{L}(\omega)$ is an isomorphism for $0 \leq a \leq 2$, $0 < |\alpha_1 + \alpha_2| < 2$, and $0 < d < 2 \arctan \sqrt{4(\alpha_1 + \alpha_2)^{-2} - 1}$. In this case, Theorem 2.2 implies that the operator \mathcal{L} is also an isomorphism.

The proof comprises three parts.

1. We prove that the equation

$$\mathcal{A}_\alpha w = f_0 \tag{2.10}$$

has a unique solution for any $f_0 \in L_2(\theta)$, where $\mathcal{A}_\alpha : D(\mathcal{A}_\alpha) \subset L_2(\theta) \rightarrow L_2(\theta)$ is the linear bounded operator given by

$$\mathcal{A}_\alpha w = -\Delta w + w, \quad w \in D(\mathcal{A}_\alpha) = \{w \in W_\alpha^1(\theta) : -\Delta w + w \in L_2(\theta)\},$$

$$\begin{aligned} W_\alpha^1(\theta) = \{w \in W^1(\theta) : & w(\varphi, r)|_{\gamma_1} - \alpha_1 w(\varphi + d/2, r)|_{\gamma_1} = 0, \\ & w(\varphi, r)|_{\gamma_2} - \alpha_2 w(\varphi - d/2, r)|_{\gamma_2} = 0\}. \end{aligned}$$

To prove the unique solvability of Eq. (2.10), we reduce it to an elliptic functional differential equation.

2. We show that each solution of Eq. (2.10) belongs to $H_1^2(\theta \cap B_R)$ for all $R > 0$.
3. We prove that the equation

$$\mathcal{L}(\omega)V = f \tag{2.11}$$

has a unique solution for any $f = (f_0, f_1, f_2) \in \mathcal{E}_a^0(\theta, \gamma)$.

1. Let us prove that Eq. (2.10) has a unique solution $w \in D(\mathcal{A}_\alpha)$ for any $f_0 \in L_2(\theta)$. To do this, we reduce Eq. (2.10) to a functional differential equation.

- 1a. Consider the functional operator $\mathcal{R} : L_2(\mathbb{R}^2) \rightarrow L_2(\mathbb{R}^2)$ given by

$$\mathcal{R}u = u(\varphi, r) + \alpha_1 u(\varphi + d/2, r) + \alpha_2 u(\varphi - d/2, r).$$

Let $I_\theta : L_2(\theta) \rightarrow L_2(\mathbb{R}^2)$ denote the operator which extends a function defined on θ to \mathbb{R}^2 by zero and $P_\theta : L_2(\mathbb{R}^2) \rightarrow L_2(\theta)$ the operator which restricts a function defined on \mathbb{R}^2 to θ . Set

$$\mathcal{R}_\theta = P_\theta \mathcal{R} I_\theta.$$

We claim that the operator \mathcal{R}_θ has the bounded inverse

$$\mathcal{R}_\theta^{-1} = P_\theta \mathcal{R}' I_\theta,$$

where

$$\mathcal{R}'u = (u(\varphi, r) - \alpha_1 u(\varphi + d/2, r) - \alpha_2 u(\varphi - d/2, r))/(1 - \alpha_1 \alpha_2),$$

provided that $\alpha_1 \alpha_2 \neq 1$ (which is true because $|\alpha_1 + \alpha_2| < 2$). Indeed,

$$\begin{aligned} \mathcal{R}_\theta u &= u(\varphi, r) + \alpha_1 u(\varphi + d/2, r) \quad \text{for } y \in \theta_1 = \{y \in \mathbb{R}^2 : 0 < \varphi < d/2, 0 < r\}, \\ \mathcal{R}_\theta u &= u(\varphi, r) + \alpha_2 u(\varphi - d/2, r) \quad \text{for } y \in \theta_2 = \{y \in \mathbb{R}^2 : d/2 < \varphi < d, 0 < r\}. \end{aligned} \tag{2.12}$$

Therefore,

$$\begin{aligned} \mathcal{R}_\theta^{-1} \mathcal{R}_\theta u &= (u(\varphi, r) + \alpha_1 u(\varphi + d/2, r) - \alpha_1 u(\varphi + d/2, r) - \alpha_1 \alpha_2 u(\varphi, r))/(1 - \alpha_1 \alpha_2) \\ &= u(\varphi, r) \quad \text{for } y \in \theta_1, \end{aligned}$$

$$\begin{aligned} \mathcal{R}_\theta^{-1} \mathcal{R}_\theta u &= (u(\varphi, r) + \alpha_2 u(\varphi - d/2, r) - \alpha_2 u(\varphi - d/2, r) - \alpha_1 \alpha_2 u(\varphi, r))/(1 - \alpha_1 \alpha_2) \\ &= u(\varphi, r) \quad \text{for } y \in \theta_2, \end{aligned}$$

which implies $\mathcal{R}_\theta^{-1}\mathcal{R}_\theta u(y) = u(y)$ for $y \in \theta$. Similarly, we obtain $\mathcal{R}_\theta\mathcal{R}_\theta^{-1}u(y) = u(y)$ for $y \in \theta$. Moreover, by using the same arguments as in Theorem 8.1 in [28, Chap. 2, Sec. 8], one can verify that the operators

$$\mathcal{R}_\theta : \dot{W}^1(\theta) \rightarrow W_\alpha^1(\theta), \quad \mathcal{R}_\theta : \dot{W}^1(\theta \cap B_R) \rightarrow W_\alpha^1(\theta \cap B_R)$$

are isomorphisms for all $R > 0$, where

$$\dot{W}^1(\theta) = \{u \in W^1(\theta) : u|_{\gamma_1} = 0, u|_{\gamma_2} = 0\},$$

$$\dot{W}^1(\theta \cap B_R) = \{u \in W^1(\theta \cap B_R) : u|_{\gamma_1} = 0, u|_{\gamma_2} = 0\},$$

$$W_\alpha^1(\theta \cap B_R) = \{w \in W^1(\theta \cap B_R) :$$

$$w(\varphi, r)|_{\gamma_1} - \alpha_1 w(\varphi + d/2, r)|_{\gamma_1} = 0, w(\varphi, r)|_{\gamma_2} - \alpha_2 w(\varphi - d/2, r)|_{\gamma_2} = 0\}.$$

1b. It follows from what has been proved in part 1a that Eq. (2.10) is equivalent to the equation

$$\mathcal{A}_\mathcal{R}u = f_0, \quad (2.13)$$

where $\mathcal{A}_\mathcal{R} : D(\mathcal{A}_\mathcal{R}) \subset L_2(\theta) \rightarrow L_2(\theta)$ is the unbounded operator given by

$$\mathcal{A}_\mathcal{R}u = (-\Delta + I)\mathcal{R}_\theta u, \quad u \in D(\mathcal{A}_\mathcal{R}) = \{u \in \dot{W}^1(\theta) : (-\Delta + I)\mathcal{R}_\theta u \in L_2(\theta)\},$$

and I stands for the identity operator in $L_2(\theta)$.

Similarly to Theorem 10.1 in [28, Chap. 2, Sec. 10], one can show that Eq. (2.13) has a unique solution for any $f_0 \in L_2(\theta)$. However, for the reader's convenience, we prefer to give the proof here.

Consider the sesquilinear form $b_\mathcal{R}[u, v]$ with the domain $D(b_\mathcal{R}) = \dot{W}^1(\theta)$ given by

$$b_\mathcal{R}[u, v] = \int_\theta \left(\sum_{i=1,2} (\mathcal{R}_\theta u)_{y_i} \overline{v_{y_i}} + \mathcal{R}_\theta u \bar{v} \right) dy. \quad (2.14)$$

It is clear that

$$\mathcal{R}_\theta u_{y_i} = (\mathcal{R}_\theta u)_{y_i} \quad \text{for} \quad u \in \dot{W}^1(\theta). \quad (2.15)$$

It follows from the Schwarz inequality and from (2.15) that

$$|b_\mathcal{R}[u, v]| \leq k_1 \|u\|_{\dot{W}^1(\theta)} \|v\|_{\dot{W}^1(\theta)}, \quad (2.16)$$

where $k_1 > 0$ does not depend on u and v .

Introduce the isomorphism $\mathcal{U} : L_2(\theta) \rightarrow L_2(\theta_1) \times L_2(\theta_1)$ by the formula

$$(\mathcal{U}u)_i(y) = u(\varphi + (i-1)d/2, r), \quad y \in \theta_1, \quad i = 1, 2.$$

Let $R_1 = \begin{pmatrix} 1 & \alpha_1 \\ \alpha_2 & 1 \end{pmatrix}$. One can directly verify that

$$R_\theta u = \mathcal{U}^{-1} R_1 \mathcal{U} u = \mathcal{U}^* R_1 \mathcal{U} u. \quad (2.17)$$

The symmetric part of the matrix R_1 has the form

$$(R_1 + R_1^*)/2 = \begin{pmatrix} 1 & (\alpha_1 + \alpha_2)/2 \\ (\alpha_1 + \alpha_2)/2 & 1 \end{pmatrix}.$$

Since $|\alpha_1 + \alpha_2| < 2$, it follows that the matrix $(R_1 + R_1^*)/2$ is positively definite. Therefore, using (2.15) and (2.17), we obtain

$$\begin{aligned} \operatorname{Re} b_\mathcal{R}[u, u] &= \int_{\theta_1} \left\{ \sum_i \left(\frac{(R_1 + R_1^*)}{2} (\mathcal{U}u_{y_i}, \mathcal{U}u_{y_i}) \right)_{\mathbb{C}^2} + \left(\frac{(R_1 + R_1^*)}{2} \mathcal{U}u, \mathcal{U}u \right)_{\mathbb{C}^2} \right\} dy \\ &\geq k_2 \int_{\theta_1} \left\{ \sum_i (\mathcal{U}u_{y_i}, \mathcal{U}u_{y_i})_{\mathbb{C}^2} + (\mathcal{U}u, \mathcal{U}u)_{\mathbb{C}^2} \right\} dy = k_2 \|u\|_{\dot{W}^1(\theta)}^2, \end{aligned} \quad (2.18)$$

where $k_2 > 0$ does not depend on u .

Inequalities (2.16) and (2.18) imply that $b_\mathcal{R}$ is a closed sectorial form on $L_2(\theta)$, with the domain $D(b_\mathcal{R}) = \dot{W}^1(\theta)$ and vertex $k_2 > 0$ (see [13, Chap. 6]). It follows from the first representation theorem (see [13, Chap. 6, Sec. 2]) that the m -sectorial operator $\mathcal{A}_\mathcal{R}$ associated with the form $b_\mathcal{R}$ has a bounded inverse $\mathcal{A}_\mathcal{R}^{-1} : L_2(\theta) \rightarrow \dot{W}^1(\theta)$.

Thus, we have proved that Eq. (2.10) has a unique solution $w = \mathcal{R}_\theta \mathcal{A}_\mathcal{R}^{-1} f_0 \in D(\mathcal{A}_\alpha)$ for any $f_0 \in L_2(\theta)$.

2. We now prove that, if $w \in D(\mathcal{A}_\alpha)$ is a solution of Eq. (2.10), then $w \in H_1^2(\theta \cap B_R(0))$ for any $R > 0$.

2a. Denote $\theta^{sj} = \theta \cap \{2^{s-j} < |y| < 2^{s+j}\}$, $\gamma_\rho^{sj} = \gamma_\rho \cap \{2^{s-j} < |y| < 2^{s+j}\}$, where $s = 0, \pm 1, \pm 2, \dots$; $\rho, j = 1, 2, 3$; $\gamma_3 = \{y \in \mathbb{R}^2 : \varphi = d/2, 0 < r\}$.

We prove that $w \in W^2(\theta^{s3})$ for any s . By theorem on interior smoothness (see, e.g., Theorem 3.2 in [19, Chap. 2, Sec. 3]), we have $w|_{\gamma_3^{s3}} \in W^{3/2}(\gamma_3^{s3})$. Since

$$w(\varphi, r)|_{\gamma_1} = \alpha_1 w(\varphi + d/2, r)|_{\gamma_1}, \quad w(\varphi, r)|_{\gamma_2} = \alpha_2 w(\varphi - d/2, r)|_{\gamma_2},$$

it follows that

$$w|_{\gamma_1^{s3}} \in W^{3/2}(\gamma_1^{s3}), \quad w|_{\gamma_2^{s3}} \in W^{3/2}(\gamma_2^{s3}).$$

Therefore, using a theorem on smoothness of solutions of local boundary-value problems in bounded domains (see e.g., Theorem 8.2 in [19, Chap. 2, Sec. 8]), we obtain $w \in W^2(\theta^{s3})$. Since $\theta^{s3} = \theta^{s-1,2} \cup \theta^{s+1,2}$, it follows that $w \in W^2(\theta^{s3})$ for any s .

2b. Let us prove that

$$\|w\|_{W^2(\theta^{01})} \leq k_3(\|-\Delta w\|_{L_2(\theta^{03})} + \|w\|_{W^1(\theta^{03})}), \quad (2.19)$$

where $k_3, k_4, \dots > 0$ do not depend on w .

To do this, we denote $\theta_3^{02} = \{y \in \theta^{02} : d/4 < \varphi < 3d/4\}$ and consider a function $\xi_0 \in C_0^\infty(\theta^{03})$ such that $\xi_0(y) = 1$ for $y \in \theta_3^{02}$.

Using the a priori estimate for solutions of local elliptic problems (see, e.g., Theorem 8.2 in [19, Chap. 2, Sec. 8] and Theorem 9.1 in [19, Chap. 2, Sec. 9]) and Leibniz' formula, we have

$$\begin{aligned} \|w\|_{\gamma_3^{02}} \|W^{3/2}(\gamma_3^{02})\| &\leq \|w\|_{W^2(\theta_3^{02})} \leq \|\xi_0 w\|_{W^2(\theta^{03})} \\ &\leq k_4 \|\Delta(\xi_0 w)\|_{L_2(\theta^{03})} \leq k_5(\|-\Delta w\|_{L_2(\theta^{03})} + \|w\|_{W^1(\theta^{03})}). \end{aligned} \quad (2.20)$$

Introduce a function $\xi_1(r) \in C_0^\infty(0, +\infty)$ such that $\xi_1(r) = 1$ for $2^{-1} \leq r \leq 2$ and $\text{supp } \xi_1 \subset (2^{-2}, 2^2)$. Applying Theorem 8.2 in [19, Chap. 2, Sec. 8] and Theorem 9.1 in [19, Chap. 2, Sec. 9] again and using (2.20), we obtain

$$\begin{aligned} \|w\|_{W^2(\theta^{01})} &\leq \|\xi_1 w\|_{W^2(\theta^{02})} \leq k_6 \left(\|\Delta(\xi_1 w)\|_{L_2(\theta^{02})} + \sum_{\rho=1,2} \|(\xi_1 w)|_{\gamma_\rho^{02}}\|_{W^{3/2}(\gamma_\rho^{02})} \right) \\ &\leq k_7(\|-\Delta w\|_{L_2(\theta^{03})} + \|w\|_{W^1(\theta^{03})}). \end{aligned}$$

Thus, inequality (2.19) is proved.

2c. Now we prove that $w \in H_1^2(\theta \cap B_R)$ for any $R > 0$. It follows from part 2a of the proof that $w \in H_1^2(\theta^{sj})$ (we set $\rho(y) = |y|$ in the definition of the space $H_1^2(\theta^{sj})$). Set $y' = 2^{-s}y$. Clearly, $y' \in \theta^{0j}$ for $y \in \theta^{sj}$. Therefore, using the fact that $2^{s-1} < r < 2^{s+1}$ for $y \in \theta^{s1}$, letting $w^s(y') = w(2^s y')$ and applying inequality (2.19), we obtain

$$\begin{aligned} \|w\|_{H_1^2(\theta^{s1})} &\leq k_8 \sum_{|\alpha| \leq 2} 2^{2s(1-2+|\alpha|)} \int_{\theta^{s1}} |D_y^\alpha w(y)|^2 dy = k_8 \sum_{|\alpha| \leq 2} \int_{\theta^{01}} |D_{y'}^\alpha w^s(y')|^2 dy' \\ &\leq k_9 \left(\|\Delta_{y'} w^s(y')\|_{L_2(\theta^{03})}^2 + \sum_{|\alpha| \leq 1} \|D_{y'}^\alpha w^s(y')\|_{L_2(\theta^{03})}^2 \right) \\ &= k_9 \left(2^{2s} \|\Delta_y w(y)\|_{L_2(\theta^{s3})}^2 + \sum_{|\alpha| \leq 1} 2^{2s(0-1+|\alpha|)} \|D_y^\alpha w(y)\|_{L_2(\theta^{s3})}^2 \right), \end{aligned} \quad (2.21)$$

where $k_7, k_8, \dots > 0$ do not depend on w and s . It follows from (2.21) that

$$\|w\|_{H_1^2(\theta^{s1})} \leq k_{10}(\|-\Delta w + w\|_{L_2(\theta^{s3})}^2 + \|w\|_{H_0^1(\theta^{s3})}) \quad (2.22)$$

for $s \leq [\log_2 R]$.

Now we claim that

$$w \in H_0^1(\theta \cap B_{8R}). \quad (2.23)$$

Indeed,

$$w \in W_\alpha^1(\theta \cap B_{8R}) \quad (2.24)$$

by assumption. Therefore, $\mathcal{R}_\theta^{-1}w \in \dot{W}^1(\theta \cap B_{8R})$ because the operator $\mathcal{R}_\theta : \dot{W}^1(\theta \cap B_{8R}) \rightarrow W_\alpha^1(\theta \cap B_{8R})$ is an isomorphism. By Lemma 4.8 in [15], $\dot{W}^1(\theta \cap B_{8R}) \subset H_0^1(\theta \cap B_{8R})$, which implies that $\mathcal{R}_\theta^{-1}w \in H_0^1(\theta \cap B_{8R})$. Therefore, using (2.12), we have

$$w \in H_0^1(\theta_1 \cap B_{8R}), \quad w \in H_0^1(\theta_2 \cap B_{8R}).$$

Combining these relations with (2.24) yields (2.23).

Summing inequalities (2.22) with respect to $s \leq [\log_2 R]$ and taking into account relation (2.23), we obtain

$$\|w\|_{H_1^2(\theta \cap B_R)} \leq k_{11}(\|\Delta w + w\|_{L_2(\theta \cap B_{8R})}^2 + \|w\|_{H_0^1(\theta \cap B_{8R})}).$$

Thus, we have proved that $w \in H_1^2(\theta \cap B_R)$.

3. We finally prove that Eq. (2.11) has a unique solution $V \in E_a^2(\theta)$ for any $f \in \mathcal{E}_a^0(\theta, \gamma)$, where $0 \leq a \leq 2$.

3a. Let $w \in D(\mathcal{A}_a)$ be a solution of Eq. (2.10) with right-hand side $f_0 \in C_0^\infty(\bar{\theta} \setminus \{0\})$. It is easy to check that the strip $-1 \leq \operatorname{Im} \lambda \leq 1$ contains no eigenvalues of the operator-valued function $\hat{\mathcal{L}}(\lambda)$ for $0 < d < 2 \arctan \sqrt{4(\alpha_1 + \alpha_2)^{-2} - 1}$ (see [9, Sec. 9.1]). On the other hand, $w \in H_1^2(\theta \cap B_1)$ by the above, and the inequalities $-1 \leq a + 1 - 2 \leq 1$ hold. Therefore, by Lemma 3.2 in [24] concerning the asymptotic behavior of solutions of nonlocal elliptic problems in plane angles, we have $w \in H_a^2(\theta \cap B_1)$.

3b. Now let us prove that the equation

$$\mathcal{L}(\omega)w = (F_0, 0, 0) \tag{2.25}$$

has a solution $w \in E_a^2(\theta)$ for any $F_0 \in E_a^0(\theta)$.

Repeating the arguments from the proof of inequality (2.4) in [22, Chap. 6, Sec. 2], one can see that a solution $w \in D(\mathcal{A}_a)$ of Eq. (2.10) with right-hand side $f_0 \in C_0^\infty(\bar{\theta} \setminus \{0\})$ belongs to $E_a^2(\theta \setminus B_{1/2})$. Combining this fact with part 3a of our proof yields $w \in E_a^2(\theta)$. Since the line $\operatorname{Im} \lambda = a + 1 - 2$ contains no eigenvalues of the operator-valued function $\hat{\mathcal{L}}(\lambda)$, it follows from Theorem 2.1 that the set of functions $F_0 \in E_a^0(\theta)$ for which Eq. (2.25) has a solution is closed in $E_a^0(\theta)$. On the other hand, the set $C_0^\infty(\bar{\theta} \setminus \{0\})$ is dense in $E_a^0(\theta)$. Therefore, Eq. (2.25) has a solution $w \in E_a^2(\theta)$ for any $F_0 \in E_a^0(\theta)$.

3c. Let us show that $\mathcal{R}(\mathcal{L}(\omega)) = \mathcal{E}_a^0(\theta, \gamma)$. Take functions $U_\rho \in E_a^2(\theta)$ such that $U_\rho|_{\gamma_\rho} = f_\rho$, $\rho = 1, 2$. Consider cut-off functions $\eta_\rho(\varphi) \in C^\infty[0, d]$ such that $\eta_1(\varphi) = 1$ for $0 \leq \varphi \leq d/4$, $\eta_1(\varphi) = 0$ for $d/3 \leq \varphi \leq d$ and $\eta_2(\varphi) = 1$ for $3d/4 \leq \varphi \leq d$, $\eta_2(\varphi) = 0$ for $0 \leq \varphi \leq 2d/3$. Then Eq. (2.11) is equivalent to Eq. (2.25), where $F_0 = \Delta U - U + f_0$, $U = \eta_1 U_1 + \eta_2 U_2 \in E_a^2(\theta)$, and $w = V - U$. It is proved in part 3b that Eq. (2.25) has a solution $w \in E_a^2(\theta)$ for any $F_0 \in E_a^0(\theta)$. Therefore, Eq. (2.11) has a solution $V = w + U \in E_a^2(\theta)$ for any $f \in \mathcal{E}_a^0(\theta, \gamma)$.

3d. It remains to prove that $\mathcal{N}(\mathcal{L}(\omega)) = \{0\}$. Let $w \in E_a^2(\theta)$ be a solution of Eq. (2.25) with $F_0 = 0$. Using the same arguments as in part 3a of this proof, we have $w \in H_1^2(\theta \cap B_1) \subset W^1(\theta \cap B_1)$. On the other hand, $E_a^2(\theta \setminus \overline{B_{1/2}}) \subset W^1(\theta \setminus \overline{B_{1/2}})$ because $a \geq 0$. Therefore, $w \in W^1(\theta)$, and $w = 0$ by part 1 of this proof.

Note that Example 2.1 was earlier studied by another method in [5, Sec. 10]. The approach proposed in [5] is based on the Green formulas for nonlocal elliptic problems (see [5]) and on the interrelation between nonlocal elliptic problems and boundary-value problems for functional differential equations (see [28]). Other examples of nonlocal elliptic problems generalizing problem (2.8), (2.9) and being uniquely solvable in dihedral angles are constructed in [23].

2.5

Given a point $g \in \mathcal{K}_2$, we consider the linear bounded operator

$$\mathcal{L}_g(\omega) : E_a^{l+2m}(\mathbb{R}_+^2) \rightarrow \mathcal{E}_a^l(\mathbb{R}_+^2, \gamma)$$

given by

$$\mathcal{L}_g(\omega)V = (A(D_y, \omega)V(y), B_{i\mu 0}(D_y, \omega)V(y)|_{\mathbb{R}_-}, B_{i\mu 0}(D_y, \omega)V(y)|_{\mathbb{R}_+}), \tag{2.26}$$

where

$$\mathcal{E}_a^l(\mathbb{R}_+^2, \gamma) = E_a^l(\mathbb{R}_+^2) \times E_a^{l+2m-m_{i\mu}-1/2}(\mathbb{R}_-) \times E_a^{l+2m-m_{i\mu}-1/2}(\mathbb{R}_+),$$

$$\mathbb{R}_+^2 = \{y \in \mathbb{R}^2 : |\varphi| < \pi/2\}, \mathbb{R}_\pm = \{y \in \mathbb{R}^2 : \varphi = \pm\pi/2\}, \omega \in S^{n-3}, \text{ cf. (1.16).}$$

We write the operators $A(D_y, 0)$ and $B_{i\mu 0}(D_y, 0)$ in the polar coordinates:

$$A(D_y, 0) = r^{-2m} \hat{A}(\varphi, D_\varphi, rD_r), \quad B_{i\mu 0}(D_y, 0) = r^{-m_{i\mu}} \hat{B}_{i\mu 0}(\varphi, D_\varphi, rD_r).$$

Consider the analytic operator-valued function

$$\hat{\mathcal{L}}_g(\lambda) : W^{l+2m}(-\pi/2, \pi/2) \rightarrow \mathcal{W}^l[-\pi/2, \pi/2]$$

given by

$$\hat{\mathcal{L}}_g(\lambda)w = (\hat{A}(\varphi, D_\varphi, \lambda)w, \hat{B}_{i\mu 0}(\varphi, D_\varphi, \lambda)w|_{\varphi=-\pi/2}, \hat{B}_{i\mu 0}(\varphi, D_\varphi, \lambda)w|_{\varphi=\pi/2}), \quad (2.27)$$

where $\mathcal{W}^l[-\pi/2, \pi/2] = W^l(-\pi/2, \pi/2) \times \mathbb{C} \times \mathbb{C}$, cf. (2.2).

It follows from [2, 4] that there exists a finite-meromorphic operator-valued function $\hat{\mathcal{R}}_g(\lambda) : \mathcal{W}^l[-\pi/2, \pi/2] \rightarrow \mathcal{W}^{l+2m}(-\pi/2, \pi/2)$ such that $\hat{\mathcal{L}}_g^{-1}(\lambda) = \hat{\mathcal{R}}_g(\lambda)$ for any λ which is not a pole of $\hat{\mathcal{R}}_g(\lambda)$. Moreover, if $\lambda_0 = \mu_0 + i\nu_0$ is a pole of $\hat{\mathcal{R}}_g(\lambda)$, then λ_0 is an eigenvalue of $\hat{\mathcal{L}}_g(\lambda)$, and there exists a $\delta > 0$ such that the set $\{\lambda \in \mathbb{C} : 0 < |\operatorname{Im} \lambda - \nu_0| < \delta\}$ contains no eigenvalues of $\hat{\mathcal{L}}_g(\lambda)$.

The following theorem establishes a connection between the operators \mathcal{L}_g and $\mathcal{L}_g(\omega)$ (see Theorem 2.1 in [22, Chap. 6, Sec. 2]).

Theorem 2.3. *Let Conditions 1.1 and 1.2 hold. Assume that the line $\operatorname{Im} \lambda = a + 1 - l - 2m$ contains no eigenvalues of $\hat{\mathcal{L}}_g(\lambda)$ and $\dim \mathcal{N}(\mathcal{L}_g(\omega)) = \operatorname{codim} \mathcal{R}(\mathcal{L}_g(\omega)) = 0$ for any $\omega \in S^{n-3}$. Then the operator $\mathcal{L}_g : \mathcal{H}_a^{l+2m}(\mathbb{R}_+^n) \rightarrow \mathcal{H}_a^l(\mathbb{R}_+^n, \Gamma)$ is an isomorphism.*

3 Local Elliptic Problems in $\mathbb{R}^n \setminus \mathcal{P}$

3.1

In Secs. 3.1 and 3.2, we recall some known results on the solvability of elliptic problems in $\mathbb{R}^2 \setminus \{0\}$. These results are adopted from [6, 24]; they will be applied to the investigation of local elliptic problems in $\mathbb{R}^n \setminus \mathcal{P}$, $n \geq 3$, see Secs. 3.3–3.5.

Let A be a properly elliptic homogeneous operator with constant complex coefficients, given by

$$A = A(D_y) = \sum_{|\alpha|=2m} a_\alpha D_y^\alpha.$$

The operator $A : H_a^{l+2m}(\mathbb{R}^2) \rightarrow H_a^l(\mathbb{R}^2)$ is bounded for any fixed integer $l \geq 0$. We consider the equation

$$Av = f_0(y), \quad y \in \mathbb{R}^2 \setminus \{0\}, \quad (3.1)$$

where $f_0 \in H_a^l(\mathbb{R}^2)$.

Write the operator $A(D_y)$ in the polar coordinates,

$$A(D_y) = r^{-2m} \hat{A}(\varphi, D_\varphi, rD_r) = r^{-2m} \sum_{\alpha_1 + \alpha_2 \leq 2m} a_{\alpha_1 \alpha_2}(\varphi) D_\varphi^{\alpha_1} (rD_r)^{\alpha_2},$$

where $a_{\alpha_1 \alpha_2} \in C_{2\pi}^\infty[0, 2\pi]$, $C_{2\pi}^\infty[0, 2\pi]$ is the set of functions defined on the interval $[0, 2\pi]$ such that their 2π -periodic extensions are infinitely differentiable on \mathbb{R} .

Setting $\tau = \ln r$, we infer from (3.1)

$$\hat{A}(\varphi, D_\varphi, D_\tau)v = F_0(\varphi, \tau), \quad 0 < \varphi < 2\pi, \quad -\infty < \tau < \infty, \quad (3.2)$$

$$D_\varphi^j v|_{\varphi=0} = D_\varphi^j v|_{\varphi=2\pi}, \quad -\infty < \tau < \infty, \quad j = 0, \dots, l + 2m - 1, \quad (3.3)$$

where $D_\tau = -i\partial/\partial\tau$, $F_0(\varphi, \tau) = e^{2m\tau} f_0(\varphi, \tau)$, $D_\varphi^j F_0|_{\varphi=0} = D_\varphi^j F_0|_{\varphi=2\pi}$, $j = 1, \dots, l - 1$.

By using the Fourier transform with respect to τ , we obtain from relations (3.2) and (3.3)

$$\hat{A}(\varphi, D_\varphi, \lambda)\hat{v}(\varphi, \lambda) = \hat{F}_0(\varphi, \lambda), \quad 0 < \varphi < 2\pi, \quad (3.4)$$

$$D_\varphi^j \hat{v}|_{\varphi=0} = D_\varphi^j \hat{v}|_{\varphi=2\pi}, \quad j = 0, \dots, l + 2m - 1. \quad (3.5)$$

Denote by $W_{2\pi}^k(0, 2\pi)$ the closure of the set $C_{2\pi}^\infty[0, 2\pi]$ in the space $W^k(0, 2\pi)$. Consider the operator-valued function $\hat{A}(\lambda) : W_{2\pi}^{l+2m}(0, 2\pi) \rightarrow W_{2\pi}^l(0, 2\pi)$ given by

$$\hat{A}(\lambda)w = \hat{A}(\varphi, D_\varphi, \lambda)w(\varphi).$$

It follows from [24, Sec. 1] that there exists a finite-meromorphic operator-valued function $\hat{\mathcal{R}}(\lambda) : W_{2\pi}^l(0, 2\pi) \rightarrow W_{2\pi}^{l+2m}(0, 2\pi)$ such that $\hat{A}^{-1}(\lambda) = \hat{\mathcal{R}}(\lambda)$ for any λ which is not a pole of $\hat{\mathcal{R}}(\lambda)$. Moreover, if $\lambda_0 = \mu_0 + i\nu_0$ is a pole of $\hat{\mathcal{R}}(\lambda)$, then λ_0 is an eigenvalue of $\hat{A}(\lambda)$, and there exists a $\delta > 0$ such that the set $\{\lambda \in \mathbb{C} : 0 < |\operatorname{Im} \lambda - \nu_0| < \delta\}$ contains no eigenvalues of $\hat{A}(\lambda)$.

The following result is proved in [24, Sec. 1].

Lemma 3.1. *Assume that the line $\operatorname{Im} \lambda = a + 1 - l - 2m$ contains no eigenvalues of the operator-valued function $\hat{A}(\lambda)$. Then Eq. (3.1) has a unique solution $v \in H_a^{l+2m}(\mathbb{R}^2)$ for any $f_0 \in H_a^l(\mathbb{R}^2)$ and*

$$\|v\|_{H_a^{l+2m}(\mathbb{R}^2)} \leq c \|f_0\|_{H_a^l(\mathbb{R}^2)}, \quad (3.6)$$

where $c > 0$ does not depend on f_0 .

3.2

Now we consider the asymptotic behavior of solutions of elliptic problems in $\mathbb{R}^2 \setminus \{0\}$. Let $l_1, l_2 \geq 0$ be integers, and let $a_1, a_2 \in \mathbb{R}$ be such that

$$h_2 = a_2 + 1 - l_2 - 2m < a_1 + 1 - l_1 - 2m = h_1.$$

By the above properties of the operator-valued function $\hat{A}(\lambda)$, the strip $h_2 < \operatorname{Im} \lambda < h_1$ contains finitely many eigenvalues λ_j , $j = 1, \dots, J$, of $\hat{A}(\lambda)$. Let q_j be the geometrical multiplicity of the eigenvalue λ_j . Denote by

$$\{\psi_j^{0,q}(\varphi), \dots, \psi_j^{p_{jq}-1,q}(\varphi)\}, \quad q = 1, \dots, q_j, \quad (3.7)$$

a canonical system of Jordan chains corresponding to the eigenvalue λ_j , where $p_{j1} \geq p_{j2} \geq \dots \geq p_{jq_j}$ are the ranks of the eigenvectors $\psi_j^{0,1}(\varphi), \dots, \psi_j^{0,q_j}(\varphi)$, respectively, see [4, Sec. 1]. It is known that the Jordan chain (3.7) satisfies the equations

$$\sum_{s=0}^p \frac{1}{s!} \partial_\lambda^s \hat{A}(\lambda_j) \psi_j^{p-s,q}(\varphi) = 0, \quad p = 0, \dots, p_{jq} - 1, \quad (3.8)$$

where $\partial_\lambda^s = \partial^s / \partial \lambda^s$.

Lemma 3.2. *Let $u \in H_{a_1}^{l_1+2m}(\mathbb{R}^2)$ be a solution of Eq. (3.1), and let $f_0 \in H_{a_1}^{l_1}(\mathbb{R}^2) \cap H_{a_2}^{l_2}(\mathbb{R}^2)$. Suppose that the line $\operatorname{Im} \lambda = h_2$ contains no eigenvalues of the operator-valued function $\hat{A}(\lambda)$. Then*

$$v(y) = \sum_{j=1}^J \sum_{q=1}^{q_j} \sum_{k=0}^{p_{jq}-1} \alpha_j^{kq} v_j^{kq}(y) + w(y), \quad y \in \mathbb{R}^2 \setminus \{0\}; \quad (3.9)$$

here

$$v_j^{kq}(y) = r^{i\lambda_j} \sum_{n=0}^k \frac{1}{n!} (i \ln r)^n \psi_j^{k-n,q}(\varphi), \quad (3.10)$$

$\alpha_j^{kq} = \alpha_j^{kq}(f_0)$ are linear bounded functionals on $H_{a_1}^{l_1}(\mathbb{R}^2) \cap H_{a_2}^{l_2}(\mathbb{R}^2)$, the function $w \in H_{a_2}^{l_2+2m}(\mathbb{R}^2)$ satisfies the equation $Aw = f_0$ and the inequality

$$\|w\|_{H_{a_2}^{l_2+2m}(\mathbb{R}^2)} \leq c \|f_0\|_{H_{a_2}^{l_2}(\mathbb{R}^2)}, \quad (3.11)$$

where $c > 0$ does not depend on f_0 .

This lemma was obtained in [24, Sec. 3] in a slightly different form. Its proof is similar to that of Theorem 1.4 in [22, Chap. 3, Sec. 1]; see also [6, Sec. 5], where the coefficients α_j^{kq} are explicitly calculated.

Remark 3.1. It is easy to see that $\psi_j^{s,q} \in C_{2\pi}^\infty[0, 2\pi]$, $j = 1, \dots, J$; $q = 1, \dots, q_j$; $s = 0, \dots, p_{jq} - 1$.

Remark 3.2. Lemma 3.2 is also valid for $h_2 \geq h_1$.

Using the same arguments as in [22, Chap. 3, Sec. 1], one can obtain the following corollaries from Lemma 3.2 (see also [6, Sec. 5]).

Corollary 3.1. *Let the conditions of Lemma 3.2 hold, and let the strip $h_2 < \operatorname{Im} \lambda < h_1$ contain no eigenvalues of $\hat{A}(\lambda)$. Then $v \in H_{a_2}^{l_2+2m}(\mathbb{R}^2)$.*

Corollary 3.2. *Let the line $\operatorname{Im} \lambda = a + 1 - l - 2m$ contain an eigenvalue of $\hat{A}(\lambda)$. Then the kernel of the operator $A : H_a^{l+2m}(\mathbb{R}^2) \rightarrow H_a^l(\mathbb{R}^2)$ is trivial, while the image of A is not closed in $H_a^l(\mathbb{R}^2)$.*

Example 3.1. We consider the equation

$$-\Delta v(y) = f_0(y), \quad y \in \mathbb{R}^2 \setminus \{0\}. \quad (3.12)$$

Introduce the operator $A : H_a^2(\mathbb{R}^2) \rightarrow H_a^0(\mathbb{R}^2)$ by the formula $Av = -\Delta v$.

Passing to the variables τ, φ and using the Fourier transform with respect to τ , we have

$$-\hat{v}_{\varphi\varphi}(\varphi, \lambda) + \lambda^2 \hat{v}(\varphi, \lambda) = \hat{F}_0(\varphi, \lambda), \quad 0 < \varphi < 2\pi, \quad (3.13)$$

$$\hat{v}|_{\varphi=0} = \hat{v}|_{\varphi=2\pi}, \quad \hat{v}_\varphi|_{\varphi=0} = \hat{v}_\varphi|_{\varphi=2\pi}, \quad (3.14)$$

where $F_0(\varphi, \tau) = e^{2\tau} f_0(\varphi, \tau)$, cf. (3.4), (3.5).

Let us study the eigenvalue problem for the corresponding operator-valued function $\hat{A}(\lambda) : W_{2\pi}^2(0, 2\pi) \rightarrow L_2(0, 2\pi)$ given by

$$\hat{A}(\lambda) \hat{v} = -\hat{v}_{\varphi\varphi} + \lambda^2 \hat{v}.$$

The general solution of the equation

$$-\hat{v}_\varphi\varphi + \lambda^2\hat{v} = 0 \quad (3.15)$$

for $\lambda \neq 0$ has the form

$$\hat{v}(\varphi) = c_1 e^{\lambda\varphi} + c_2 e^{-\lambda\varphi}. \quad (3.16)$$

Substituting this solution into (3.14), we have

$$\begin{aligned} c_1(1 - e^{\lambda 2\pi}) + c_2(1 - e^{-\lambda 2\pi}) &= 0, \\ c_1\lambda(1 - e^{\lambda 2\pi}) - c_2\lambda(1 - e^{-\lambda 2\pi}) &= 0. \end{aligned}$$

Therefore, the set of nonzero eigenvalues of $\hat{A}(\lambda)$ coincides with the set of nonzero roots of the equation

$$e^{\lambda 2\pi} = 1.$$

The nonzero roots of this equation have the form

$$\lambda_k = ik, \quad k = \pm 1, \pm 2, \dots \quad (3.17)$$

It is evident that $\lambda_0 = 0$ is also an eigenvalue of $\hat{A}(\lambda)$.

1. Let us first consider the eigenvalue $\lambda_0 = 0$. The corresponding eigenvector has the form $\psi_0^{0,1}(\varphi) = 1$ (up to factor). The geometric multiplicity of $\lambda_0 = 0$ is equal to 1.

Due to (3.8), an associated vector $\psi_0^{1,1}(\varphi)$ must satisfy the equation

$$\hat{A}(0)\psi_0^{1,1} + \partial_\lambda \hat{A}(0)\psi_0^{0,1} = 0,$$

which is equivalent to the following problem:

$$\begin{aligned} (\psi_0^{1,1})_\varphi &= 0, \quad 0 < \varphi < 2\pi, \\ \psi_0^{1,1}|_{\varphi=0} &= \psi_0^{1,1}|_{\varphi=2\pi}, \quad (\psi_0^{1,1})_\varphi|_{\varphi=0} = (\psi_0^{1,1})_\varphi|_{\varphi=2\pi}. \end{aligned}$$

Hence, $\psi_0^{1,1} = 0$ (note that an associated vector of an operator-valued function, unlike an eigenvector, can be equal to zero).

Due to (3.8), the second associated vector $\psi_0^{2,1}(\varphi)$ must satisfy the equation

$$\hat{A}(0)\psi_0^{2,1} + \partial_\lambda \hat{A}(0)\psi_0^{1,1} + \frac{1}{2}\partial_\lambda^2 \hat{A}(0)\psi_0^{0,1} = 0,$$

which is equivalent to the following problem:

$$\begin{aligned} (\psi_0^{2,1})_\varphi &= 1, \quad 0 < \varphi < 2\pi, \\ \psi_0^{2,1}|_{\varphi=0} &= \psi_0^{2,1}|_{\varphi=2\pi}, \quad (\psi_0^{2,1})_\varphi|_{\varphi=0} = (\psi_0^{2,1})_\varphi|_{\varphi=2\pi}. \end{aligned} \quad (3.18)$$

Substituting the general solution $\psi_0^{2,1}(\varphi) = c_0 + c_1\varphi + \varphi^2/2$ of Eq. (3.18) into (3.19), we obtain

$$2\pi c_1 + 4\pi^2/2 = 0, \quad 2\pi = 0.$$

This system is incompatible, and hence there does not exist a second associated vector for $\lambda_0 = 0$.

2. Consider the eigenvalue $\lambda_k = ik$, $k = \pm 1, \pm 2, \dots$. There are two linearly independent eigenvectors corresponding to λ_k ,

$$\psi_k^{0,1} = \sin k\varphi, \quad \psi_k^{0,2} = \cos k\varphi$$

(up to factor). Hence, the geometric multiplicity of λ_k equals 2.

Due to (3.8), an associated vector $\psi_k^{1,j}(\varphi)$ ($j = 1, 2$) must satisfy the equation

$$\hat{A}(ik)\psi_k^{1,j} + \partial_\lambda \hat{A}(ik)\psi_k^{0,j} = 0,$$

which is equivalent to the following problem:

$$(\psi_k^{1,j})_\varphi + k^2\psi_k^{1,j} = i2k\psi_k^{0,j}, \quad 0 < \varphi < 2\pi, \quad (3.20)$$

$$\psi_k^{1,j}|_{\varphi=0} = \psi_k^{1,j}|_{\varphi=2\pi}, \quad (\psi_k^{1,j})_\varphi|_{\varphi=0} = (\psi_k^{1,j})_\varphi|_{\varphi=2\pi}. \quad (3.21)$$

Substituting the general solution

$$\begin{aligned} \psi_k^{1,1}(\varphi) &= a_k^1 \cos k\varphi + b_k^1 \sin k\varphi - i\varphi \cos k\varphi & \text{for } j = 1, \\ \psi_k^{1,2}(\varphi) &= a_k^2 \cos k\varphi + b_k^2 \sin k\varphi + i\varphi \sin k\varphi & \text{for } j = 2 \end{aligned}$$

of Eq. (3.20) into (3.21), we have

$$\begin{aligned} 0 &= -i2\pi, \quad 0 = 0 & \text{for } j = 1, \\ 0 &= 0, \quad 0 = i2\pi k & \text{for } j = 2. \end{aligned}$$

These systems are incompatible for $k = \pm 1, \pm 2, \dots$, and hence there do not exist associated vectors for $\lambda_k = ik$, $k = \pm 1, \pm 2, \dots$.

Example 3.2. Let $v \in H_1^2(\mathbb{R}^2)$ be a solution of Eq. (3.12) with right-hand side $f_0 \in H_1^0(\mathbb{R}^2) \cap H_{-\varepsilon}^0(\mathbb{R}^2)$, where $0 < \varepsilon < 1$. We study the asymptotic behavior of the solution v . Using the results of Example 3.1 and Lemma 3.2 for $a_1 = 1, l_1 = 0, h_1 = 0$ and $a_2 = -\varepsilon, l_2 = 0, h_2 = -1 - \varepsilon$, we obtain

$$v(y) = (\alpha_1^{0,1} \sin \varphi + \alpha_1^{0,2} \cos \varphi)r + w(y) = \alpha_1^{0,1}y_2 + \alpha_1^{0,2}y_1 + w(y), \quad y \in \mathbb{R}^2 \setminus \{0\}, \quad (3.22)$$

where $w \in H_{-\varepsilon}^2(\mathbb{R}^2)$, while $\alpha_1^{0,j} = \alpha_1^{0,j}(f_0)$ are linear bounded functionals on $H_1^0(\mathbb{R}^2) \cap H_{-\varepsilon}^0(\mathbb{R}^2)$. Note that these functionals can be found in an explicit form (see [6, Sec. 5]).

3.3

We now proceed with the study of elliptic problems in $\mathbb{R}^n \setminus \mathcal{P}$ for $n \geq 3$, where

$$\mathcal{P} = \{x = (y, z) \in \mathbb{R}^n : y = 0, z \in \mathbb{R}^{n-2}\}.$$

Let

$$A = A(D_y, D_z) = \sum_{|\alpha|+|\beta|=2m} a_{\alpha\beta} D_y^\alpha D_z^\beta,$$

be a homogeneous properly elliptic operator with constant complex coefficients. We consider the equation

$$Av = f_0(x), \quad x \in \mathbb{R}^n \setminus \mathcal{P}, \quad (3.23)$$

where $f_0 \in H_a^l(\mathbb{R}^n)$. It is easy to see that the operator $A : H_a^{l+2m}(\mathbb{R}^n) \rightarrow H_a^l(\mathbb{R}^n)$ is bounded for any $a \in \mathbb{R}$ and any integer $l \geq 0$.

The main result of this section is as follows.

Theorem 3.1. *Let $a \in \mathbb{R}$, and let $l \geq 0$ be an integer. Then the operator $A : H_a^{l+2m}(\mathbb{R}^n) \rightarrow H_a^l(\mathbb{R}^n)$ is not an isomorphism.*

To prove Theorem 3.1, we first apply the Fourier transform $F_{z \rightarrow \eta}$ with respect to $z \in \mathbb{R}^{n-2}$. Then Eq. (3.23) takes the form

$$A(D_y, \eta) \tilde{v}(y, \eta) = \tilde{f}_0(y, \eta), \quad y \in \mathbb{R}^2 \setminus \{0\}, \quad \eta \in \mathbb{R}^{n-2}, \quad (3.24)$$

where

$$\tilde{v}(y, \eta) = F_{z \rightarrow \eta} v = (2\pi)^{-(n-2)/2} \int_{\mathbb{R}^{n-2}} v(y, z) e^{-i(\eta, z)} dz.$$

Denote $Y = |\eta|y$, $\omega = \eta/|\eta|$, $V(Y, \eta) = |\eta|^{2m} \tilde{v}(y, \eta)$, $F_0(Y, \eta) = \tilde{f}_0(y, \eta)$. Then Eq. (3.24) takes the following form:

$$A(D_Y, \omega) V(Y, \eta) = F_0(Y, \eta), \quad y \in \mathbb{R}^2 \setminus \{0\}, \quad \omega \in S^{n-3}. \quad (3.25)$$

Consider the linear bounded operator $A(\omega) : E_a^{l+2m}(\mathbb{R}^2) \rightarrow E_a^l(\mathbb{R}^2)$ given by

$$A(\omega)V = A(D_Y, \omega)V(Y), \quad \omega \in S^{n-3}. \quad (3.26)$$

Denote by $A(0) : H_a^{l+2m}(\mathbb{R}^2) \rightarrow H_a^l(\mathbb{R}^2)$ the linear bounded operator given by

$$A(0)v = A(D_y, 0)v(y).$$

Clearly, the operator $A(0)$ is properly elliptic. Write the operator $A(0)$ in the polar coordinates,

$$A(0) = r^{-2m} \hat{A}(\varphi, D_\varphi, rD_r).$$

Consider the operator-valued function $\hat{A}(\lambda) : W_{2\pi}^{l+2m}(0, 2\pi) \rightarrow W_{2\pi}^l(0, 2\pi)$ given by

$$\hat{A}(\lambda)w = \hat{A}(\varphi, D_\varphi, \lambda)w(\varphi).$$

Spectral properties of the operator-valued function $\hat{A}(\lambda)$ are described in Sec. 3.1.

The proof of the following lemma is similar to that of Theorem 2.3 in [22, Chap. 6, Sec. 2] (see also Theorem 4.2 in [25]).

Lemma 3.3. *The operator $A(\omega) : E_a^{l+2m}(\mathbb{R}^2) \rightarrow E_a^l(\mathbb{R}^2)$ has the Fredholm property for each $\omega \in S^{n-3}$ iff the line $\text{Im } \lambda = a + 1 - l - 2m$ contains no eigenvalues of the operator-valued function $\hat{A}(\lambda)$.*

However, we prove below (see Lemma 3.10) that the operator $A(\omega) : E_a^{l+2m}(\mathbb{R}^2) \rightarrow E_a^l(\mathbb{R}^2)$ is not an isomorphism for $a \in \mathbb{R}$, $l \geq 0$, and $\omega \in S^{n-3}$.

The following result is valid (see [25]).

Lemma 3.4. *The operator $A : H_a^{l+2m}(\mathbb{R}^n) \rightarrow H_a^l(\mathbb{R}^n)$ is an isomorphism iff the operator $A(\omega) : E_a^{l+2m}(\mathbb{R}^2) \rightarrow E_a^l(\mathbb{R}^2)$ is an isomorphism.*

Combining Lemma 3.4 with the fact that $A(\omega)$ is not an isomorphism allows us to prove Theorem 3.1. Thus, it remains to show that the operator $A(\omega)$ is not an isomorphism for $a \in \mathbb{R}$, $l \geq 0$, and $\omega \in S^{n-3}$. To do so, we preliminarily establish a priori estimates for solutions of Eq. (3.25) and study the adjoint operators.

The proof of a priori estimates in the spaces $E_a^l(\mathbb{R}^2)$ is based on the well-known a priori estimate in Sobolev spaces (see, e.g., Theorem 15.3 in [1] and the comment following it).

Lemma 3.5. *Let $Q_1, Q_2 \subset \mathbb{R}^n$ be bounded domains such that $\overline{Q_1} \subset Q_2$. Assume that an operator*

$$\mathcal{A} = \sum_{|\alpha| \leq 2m} a_\alpha(x) D_x^\alpha$$

with infinitely differentiable coefficients $a_\alpha(x)$ is properly elliptic on $\overline{Q_2}$. Then the following estimate holds for all $u \in W^{l+2m}(Q_2)$:

$$\|u\|_{W^{l+2m}(Q_1)} \leq c(\|\mathcal{A}u\|_{W^l(Q_2)} + \|u\|_{L_2(Q_2)}), \quad (3.27)$$

where $c > 0$ depends on Q_1, Q_2 , and M ,

$$M = \max_{|\beta| \leq l_0} \max_{|\alpha| \leq 2m} \max_{x \in \overline{Q_2}} |D^\beta a_\alpha(x)|, \quad l_0 = \max(l, 1),$$

and does not depend on u .

Remark 3.3. Theorem 3.1 in [19, Chap. 2, Sec. 3] ensures the validity of estimate (3.27) with the term $\|u\|_{W^{l+2m-1}(Q_2)}$ instead of $\|u\|_{L_2(Q_2)}$ on the right-hand side. To obtain estimate (3.27), one must additionally apply the technique close to that in [21, Chap. 5].

Denote by $W_{\text{loc}}^k(\mathbb{R}^2 \setminus \{0\})$ the space of distributions v on $\mathbb{R}^2 \setminus \{0\}$ such that $\psi v \in W^k(\mathbb{R}^2)$ for all $\psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$.

Lemma 3.6. *Let $v \in W_{\text{loc}}^{l+2m}(\mathbb{R}^2 \setminus \{0\}) \cap E_{a-l-2m}^0(\{|y| < 1\})$ and $A(\omega)v \in E_a^l(\mathbb{R}^2)$ for some $\omega \in S^{n-3}$. Then $v \in E_a^{l+2m}(\mathbb{R}^2)$ and*

$$\|v\|_{E_a^{l+2m}(\mathbb{R}^2)} \leq c(\|A(\omega)v\|_{E_a^l(\mathbb{R}^2)} + \|v\|_{E_{a-l-2m}^0(\{|y| < R\})}), \quad (3.28)$$

where $R, c > 0$ do not depend on v and ω .

Proof. 1. Denote $Q_1^s = \{y \in \mathbb{R}^2 : 2^s < |y| < 2^{s+1}\}$ and $Q_2^s = \{y \in \mathbb{R}^2 : 2^{s-1} < |y| < 2^{s+2}\}$, $s = 0, \pm 1, \pm 2, \dots$. Evidently, $\overline{Q_1^s} \subset Q_2^s$. It follows from the belonging $v \in W_{\text{loc}}^{l+2m}(\mathbb{R}^2 \setminus \{0\})$ that $v \in H_a^{l+2m}(Q_2^s) \cap E_a^{l+2m}(Q_2^s)$ for any s (we set $\rho(y) = |y|$ in the definition of the spaces $H_a^{l+2m}(Q_2^s)$ and $E_a^{l+2m}(Q_2^s)$). First, we prove that

$$\|v\|_{E_a^{l+2m}(Q_1^s)}^2 \leq k_1(\|A(\omega)v\|_{E_a^l(Q_2^s)}^2 + \|v\|_{E_{a-l-2m}^0(Q_2^s)}^2), \quad s \leq 0, \quad (3.29)$$

where $k_1, k_2, \dots > 0$ do not depend on v, ω , and s .

Set $y' = 2^{-s}y$. Clearly, $y' \in Q_j^0$ for $y \in Q_j^s$, $j = 1, 2$; $s = 0, \pm 1, \pm 2, \dots$. Therefore, setting $v^s(y') = v(2^s y')$ and applying Lemma 3.5, we obtain, for $s \leq 0$,

$$\begin{aligned} \|v\|_{H_a^{l+2m}(Q_1^s)}^2 &\leq k_2 \sum_{|\alpha| \leq l+2m} 2^{2s(a-l-2m+|\alpha|)} \|D_y^\alpha v(y)\|_{L_2(Q_1^s)}^2 \\ &= k_2 \sum_{|\alpha| \leq l+2m} 2^{2s(a-l-2m+1)} \|D_{y'}^\alpha v^s(y')\|_{L_2(Q_1^0)}^2 \\ &\leq k_3 2^{2s(a-l-2m+1)} \left(\sum_{|\alpha| \leq l} \|D_{y'}^\alpha A(D_{y'}, 2^s \omega) v^s(y')\|_{L_2(Q_2^0)}^2 + \|v^s(y')\|_{L_2(Q_2^0)}^2 \right) \\ &= k_3 \left(\sum_{|\alpha| \leq l} 2^{2s(a-l+|\alpha|)} \|D_y^\alpha A(D_y, \omega) v(y)\|_{L_2(Q_2^s)}^2 + 2^{2s(a-l-2m)} \|v(y)\|_{L_2(Q_2^s)}^2 \right) \\ &\leq k_4 (\|A(D_y, \omega)v\|_{H_a^l(Q_2^s)}^2 + \|v\|_{H_{a-l-2m}^0(Q_2^s)}^2). \end{aligned}$$

The latter estimate is equivalent to (3.29).

2. To complete the proof, it remains to show that the estimate in (3.29) is also valid for $s > 0$. To do so, we apply Lemma 3.5 for $\mathcal{A} = A(D_{y'}, D_z)$, $u(y', z) = \exp(i2^s(\omega, z))v^s(y')$, and

$$Q_j = Q_j^0 \times \{z \in \mathbb{R}^{n-2} : |z_k| < j, k = 1, \dots, n-2\}, \quad j = 1, 2.$$

Then we obtain

$$\begin{aligned} \sum_{\nu=0}^{l+2m} 2^{2s\nu} \|v^s(y')\|_{W^{l+2m-\nu}(Q_1^0)}^2 \\ \leq k_5 \left(\sum_{\nu=0}^l 2^{2s\nu} \|A(D_{y'}, 2^s \omega) v^s(y')\|_{W^{l-\nu}(Q_2^0)}^2 + \|v^s(y')\|_{L_2(Q_2^0)}^2 \right). \end{aligned} \quad (3.30)$$

Since $s > 0$, it follows that inequality (3.30) is equivalent to the following:

$$2^{2s(l+2m-1)} \|v(y)\|_{W^{l+2m}(Q_1^s)}^2 \leq k_6 (2^{2s(l+2m-1)} \|A(D_y, \omega) v(y)\|_{W^l(Q_2^s)}^2 + 2^{-2s} \|v(y)\|_{L_2(Q_2^s)}^2).$$

Multiplying both sides of this inequality by $2^{2s(a-l-2m+1)}$ yields

$$\|v\|_{E_a^{l+2m}(Q_1^s)}^2 \leq k_1 (\|A(\omega) v\|_{E_a^l(Q_2^s)}^2 + 2^{2s(a-l-2m)} \|v\|_{L_2(Q_2^s)}^2), \quad s > 0. \quad (3.31)$$

Summing (3.29) and (3.31) with respect to $s = 0, -1, -2, \dots$ and $s = 1, 2, \dots$, respectively, and choosing a sufficiently large $R > 0$, we obtain (3.28). \square

Using Lemmas 3.2 and 3.6, we can prove the following result on regularity of solutions of the equation

$$A(\omega)v = f_0, \quad \omega \in S^{n-3}. \quad (3.32)$$

Lemma 3.7. *Let the closed strip bounded by the lines $\text{Im } \lambda = a_1 + 1 - l_1 - 2m$ and $\text{Im } \lambda = a_2 + 1 - l_2 - 2m$ contain no eigenvalues of the operator-valued function $\hat{A}(\lambda)$. Suppose that $v \in E_{a_1}^{l_1+2m}(\mathbb{R}^2)$ is a solution of Eq. (3.32) for some $\omega \in S^{n-3}$, with right-hand side $f_0 \in E_{a_1}^{l_1}(\mathbb{R}^2) \cap E_{a_2}^{l_2}(\mathbb{R}^2)$. Then $v \in E_{a_2}^{l_2+2m}(\mathbb{R}^2)$.*

Proof. 1. Consider a function $\eta \in C^\infty(\mathbb{R})$ such that $\eta(r) = 0$ for $r \leq 1$ and $\eta(r) = 1$ for $r \geq 2$. Denote by $[A(\omega), \eta]$ the commutator of $A(\omega)$ and η . It is clear that $\text{supp } [A(\omega), \eta]v \subset \{y \in \mathbb{R}^2 : 1 \leq |y| \leq 2\}$. Therefore,

$$A(\omega)(\eta v) = \eta f_0 + [A(\omega), \eta]v \in E_{a_1}^{l_1}(\mathbb{R}^2) \cap E_{a_2}^{l_2}(\mathbb{R}^2) \quad (3.33)$$

because $v \in W_{\text{loc}}^{l+2m}(\mathbb{R}^2 \setminus \{0\})$, where $l = \max(l_1, l_2)$, due to Theorem 3.2 in [19, Chap. 2, Sec. 3].

On the other hand, ηv vanishes near the origin, and hence $\eta v \in E_{a_2-l_2-2m}^0(\{|y| < R\})$ for any $R > 0$. Using this fact, relation (3.33), and Lemma 3.6, we conclude that $\eta v \in E_{a_2}^{l_2+2m}(\mathbb{R}^2)$.

2. Since $\text{supp } A(\omega)((1-\eta)v) \subset \{y \in \mathbb{R}^2 : |y| \leq 2\}$, we obtain (similarly to (3.33)) that

$$A(\omega)((1-\eta)v) \in H_{a_1}^{l_1}(\mathbb{R}^2) \cap H_{a_2}^{l_2}(\mathbb{R}^2). \quad (3.34)$$

Therefore, by using Lemma 3.2 and Remark 3.2, we conclude that¹ $(1-\eta)v \in H_{a_2}^{l_2+2m}(\mathbb{R}^2)$, and hence $(1-\eta)v \in E_{a_2}^{l_2+2m}(\mathbb{R}^2)$.

Thus, $v \in E_{a_2}^{l_2+2m}(\mathbb{R}^2)$. \square

3.4

In this subsection, we consider adjoint operators. Introduce the operator

$$A'(\omega) = A'(D_y, \omega) = \sum_{|\alpha|+|\beta|=2m} \overline{a_{\alpha\beta}} \omega^\beta D_y^\alpha.$$

The operator $A'(D_y, \omega)$ is formally adjoint to $A(D_y, \omega)$ with respect to the Green formula, i.e.,

$$\int_{\mathbb{R}^2} A(D_y, \omega) u \overline{v} dy = \int_{\mathbb{R}^2} u \overline{A'(D_y, \omega) v} dy, \quad \omega \in \mathbb{R}^{n-2}, \quad (3.35)$$

for all $u, v \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$.

Consider the unbounded operators

$$\begin{aligned} \mathcal{A}(\omega) : D(\mathcal{A}(\omega)) &\subset E_{b-2m}^0(\mathbb{R}^2) \rightarrow E_b^0(\mathbb{R}^2), \\ \mathcal{A}(\omega)v &= A(D_y, \omega)v, \quad v \in D(\mathcal{A}(\omega)) = E_b^{2m}(\mathbb{R}^2) \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}'(\omega) : D(\mathcal{A}'(\omega)) &\subset E_{-b}^0(\mathbb{R}^2) \rightarrow E_{2m-b}^0(\mathbb{R}^2), \\ \mathcal{A}'(\omega)v &= A'(D_y, \omega)v, \quad v \in D(\mathcal{A}'(\omega)) = E_{2m-b}^{2m}(\mathbb{R}^2). \end{aligned}$$

¹Since the operator $A(\omega)$ contains lower-order terms, one must consecutively apply Lemma 3.2 finitely many times, cf. [15, 16].

Lemma 3.8. *The operator $\mathcal{A}'(\omega)$ is adjoint to $\mathcal{A}(\omega)$ with respect to the inner product in $L_2(\mathbb{R}^2)$ for any $\omega \in S^{n-3}$.*

Proof. Denote by $\mathcal{A}^*(\omega)$ the adjoint operator for $\mathcal{A}(\omega)$. Since $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ is dense in the spaces $E_b^{2m}(\mathbb{R}^2)$ and $E_{2m-b}^{2m}(\mathbb{R}^2)$, it follows that identity (3.35) is valid for all $u \in E_b^{2m}(\mathbb{R}^2)$ and $v \in E_{2m-b}^{2m}(\mathbb{R}^2)$. Therefore, $\mathcal{A}'(\omega) \subset \mathcal{A}^*(\omega)$.

It remains to prove the inverse inclusion. Let $v \in D(\mathcal{A}^*(\omega)) \subset E_{-b}^0(\mathbb{R}^2)$. Since $\mathcal{A}^*(\omega)v \in E_{2m-b}^0(\mathbb{R}^2) \subset L_{2,\text{loc}}(\mathbb{R}^2 \setminus \{0\})$, it follows from Theorem 3.2 in [19, Chap. 2, Sec. 3] that $v \in W_{\text{loc}}^{2m}(\mathbb{R}^2 \setminus \{0\})$. Therefore, by Lemma 3.6, $v \in E_{2m-b}^{2m}(\mathbb{R}^2)$, and hence $\mathcal{A}^*(\omega) \subset \mathcal{A}'(\omega)$. \square

Consider the identity (3.35) for $\omega = 0$, substitute $u = u_1$ and $v = r^{2m-2}v_1$ into it, and set $\tau = \ln r$. Then we have

$$\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} \left(\hat{A}(\varphi, D_\varphi, D_\tau) u_1 \overline{v_1} - \overline{\hat{A}'(\varphi, D_\varphi, D_\tau - 2i(m-1)) v_1} \right) d\varphi = 0 \quad (3.36)$$

for all $u_1, v_1 \in \{u \in C_0^\infty([0, 2\pi] \times \mathbb{R}) : D_\varphi^j u|_{\varphi=0} = D_\varphi^j u|_{\varphi=2\pi}, j = 0, 1, \dots\}$, where $\hat{A}'(\varphi, D_\varphi, D_\tau)$ is defined similarly to $\hat{A}(\varphi, D_\varphi, D_\tau)$.

Consider functions $\psi, \hat{\psi} \in C_0^\infty(\mathbb{R})$ such that

$$\psi(\tau) = 0 \quad \text{for } |\tau| > 1, \quad \int_{-\infty}^{\infty} \psi(\tau) d\tau = 1,$$

$$\hat{\psi}(\tau) = 1 \quad \text{for } |\tau| < 1, \quad \hat{\psi}(\tau) = 0 \quad \text{for } |\tau| > 2.$$

Substituting $u_1 = e^{i\lambda\tau}\psi(\tau)u_2(\varphi)$ and $v_1 = e^{i\bar{\lambda}\tau}\hat{\psi}(\tau)v_2(\varphi)$ into (3.36), we obtain

$$\int_0^{2\pi} \left(\hat{A}(\varphi, D_\varphi, \lambda) u_2 \overline{v_2} - \overline{u_2 \hat{A}'(\varphi, D_\varphi, \bar{\lambda} - 2i(m-1)) v_2} \right) d\varphi = 0 \quad (3.37)$$

for all $u_2, v_2 \in C_{2\pi}^\infty[0, 2\pi]$ and $\lambda \in \mathbb{C}$.

Along with $\hat{A}(\lambda)$, we consider the operator-valued function $\hat{A}'(\lambda) : W_{2\pi}^{2m}(0, 2\pi) \rightarrow L_2(0, 2\pi)$ given by

$$\hat{A}'(\lambda)w = \hat{A}'(\varphi, D_\varphi, \lambda)w.$$

We also introduce the unbounded operators

$$\hat{A}(\lambda), \hat{A}'(\lambda) : D(\hat{A}(\lambda)) = D(\hat{A}'(\lambda)) \subset L_2(0, 2\pi) \rightarrow L_2(0, 2\pi)$$

given by

$$\begin{aligned} \hat{A}(\lambda)w &= \hat{A}(\varphi, D_\varphi, \lambda)w, & w &\in D(\hat{A}(\lambda)) = W_{2\pi}^{2m}(0, 2\pi), \\ \hat{A}'(\lambda)w &= \hat{A}'(\varphi, D_\varphi, \lambda)w, & w &\in D(\hat{A}'(\lambda)). \end{aligned}$$

Similarly to Lemma 3.8, we conclude from (3.37) that the operator $\hat{A}'(\bar{\lambda} - 2i(m-1))$ is adjoint to $\hat{A}(\lambda)$ with respect to the inner product in $L_2(0, 2\pi)$ for any $\lambda \in \mathbb{C}$. This fact and the fact that $\hat{A}(\lambda)$ is a Fredholm operator with $\text{ind } \hat{A}(\lambda) = 0$ imply:

Lemma 3.9. *A number λ is an eigenvalue of $\hat{A}(\lambda)$ iff $\bar{\lambda} - 2i(m-1)$ is an eigenvalue of $\hat{A}'(\lambda)$.*

3.5

In this subsection, using the results of Secs. 3.1–3.4, we prove the following result.

Lemma 3.10. *Let $a \in \mathbb{R}$, $l \geq 0$ be an integer, and $\omega \in S^{n-3}$. Then the operator $A(\omega) : E_a^{l+2m}(\mathbb{R}^2) \rightarrow E_a^l(\mathbb{R}^2)$ is not an isomorphism.*

We preliminarily prove two lemmas on the property of the operator $A(\omega)$ to be an isomorphism. These lemmas, together with properties of the adjoint operator, will enable us to prove Lemma 3.10 and hence Theorem 3.1.

Lemma 3.11. *Assume that the strip $a_2 + 1 - l - 2m < \text{Im } \lambda < a_1 + 1 - l - 2m$ contains no eigenvalues of the operator-valued function $\hat{A}(\lambda)$. If the operator $A(\omega)$, $\omega \in S^{n-3}$, is an isomorphism for some $a = a_0 \in (a_2, a_1)$, then it is an isomorphism for all $a \in (a_2, a_1)$.*

Proof. 1. It suffices to prove that $\mathcal{N}(A(\omega)) = \{0\}$ and $\mathcal{R}(A(\omega)) = E_b^l(\mathbb{R}^2)$ for each $a = b \in (a_2, a_1)$. It follows from Lemma 3.7 that, if $v \in \mathcal{N}(A(\omega))$ for $a = b$, $a_2 < b < a_1$, then $v \in \mathcal{N}(A(\omega))$ for $a = a_0$. Hence $v = 0$, i.e., $\mathcal{N}(A(\omega)) = \{0\}$ for $a = b$.

2. Consider Eq. (3.32) for $f_0 \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ and $a = a_0$. By assumption, this equation has a unique solution $v \in E_{a_0}^{l+2m}(\mathbb{R}^2)$. By virtue of Lemma 3.7, $v \in E_b^{l+2m}(\mathbb{R}^2)$. Lemma 3.3 implies that $\mathcal{R}(A(\omega))$ (for $a = b$) is closed in $E_b^l(\mathbb{R}^2)$. Combining this with the fact that $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ is dense in $E_b^l(\mathbb{R}^2)$ yields $\mathcal{R}(A(\omega)) = E_b^l(\mathbb{R}^2)$. \square

Lemma 3.12. *Assume that each of the lines $\text{Im } \lambda = a_2 + 1 - l - 2m$ and $\text{Im } \lambda = a_1 + 1 - l - 2m$ contains an eigenvalue of the operator-valued function $\hat{A}(\lambda)$, and let the strip $a_2 + 1 - l - 2m < \text{Im } \lambda < a_1 + 1 - l - 2m$ contain no eigenvalues of the operator-valued function $\hat{A}(\lambda)$. If the operator $A(\omega)$, $\omega \in S^{n-3}$, is an isomorphism for some $a = a_0 \in (a_2, a_1)$, then it is not an isomorphism for $a \notin (a_2, a_1)$.*

Proof. 1. Let us prove that $\dim \mathcal{N}(A(\omega)) > 0$ for $a > a_1$. Set

$$u = r^{i\lambda_0} \psi^0(\varphi),$$

where λ_0 is an eigenvalue of the operator-valued function $\hat{A}(\lambda)$ such that $\text{Im } \lambda_0 = a_1 + 1 - l - 2m$ and $\psi^0(\varphi)$ is the corresponding eigenvector. In this case,

$$A(0)u = 0. \quad (3.38)$$

Therefore,

$$A(\omega)((1 - \eta)u) = [A(0), (1 - \eta)]u + (A(\omega) - A(0))((1 - \eta)u) \equiv F,$$

where η is the same function as in the proof of Lemma 3.7.

Note that

$$(1 - \eta)u \notin E_b^{l+2m}(\mathbb{R}^2) \quad \text{for any } b \leq a_1, \quad (3.39)$$

$$(1 - \eta)u \in E_a^{l+2m}(\mathbb{R}^2) \quad \text{for any } a > a_1. \quad (3.40)$$

Clearly,

$$F \in E_b^l(\mathbb{R}^2) \quad \text{for any } b \in (a_1 - 1, +\infty). \quad (3.41)$$

By Lemma 3.11, Eq. (3.32) with the right-hand side $f_0 = F$ has a unique solution $v \in E_b^{l+2m}(\mathbb{R}^2)$, where $b \in (a_2, a_1) \cap (a_1 - 1, +\infty)$. In particular, this implies that $w = (1 - \eta)u - v$ is not the zero function due to (3.39).

Further, using the relation $v \in E_b^{l+2m}(\mathbb{R}^2) \subset E_b^0(\mathbb{R}^2)$ and taking (3.41) into account, we deduce from Lemma 3.6 that $v \in E_{b+l+2m}^{l+2m}(\mathbb{R}^2)$. Repeating these arguments finitely many times, we obtain that $v \in E_a^{l+2m}(\mathbb{R}^2)$ for any $a > a_1$. Combining this relation with (3.40), we see that $w \in E_a^{l+2m}(\mathbb{R}^2)$ for $a > a_1$, and hence $w \in \mathcal{N}(A(\omega))$ for $a > a_1$.

2. Now let $a < a_2$. If the line $\text{Im } \lambda = a + 1 - l - 2m$ contains an eigenvalue of $\hat{A}(\lambda)$, then the conclusion of this lemma follows from Lemma 3.3. Therefore, we assume that the line $\text{Im } \lambda = a + 1 - l - 2m$ contains no eigenvalues of $\hat{A}(\lambda)$. In this case, the set $\mathcal{R}(A(\omega))$ is closed both for $A(\omega) : E_{a-l}^{2m}(\mathbb{R}^2) \rightarrow E_{a-l}^0(\mathbb{R}^2)$ and for $A(\omega) : E_a^{l+2m}(\mathbb{R}^2) \rightarrow E_a^l(\mathbb{R}^2)$.

2a. First, we prove that $d = \dim \mathcal{R}(A(\omega))^\perp > 0$ for

$$A(\omega) : E_{a-l}^{2m}(\mathbb{R}^2) \rightarrow E_{a-l}^0(\mathbb{R}^2).$$

By virtue of Lemma 3.9, the lines $\text{Im } \lambda = l + 2m - a_2 + 1 - 2m$ and $\text{Im } \lambda = l + 2m - a_1 + 1 - 2m$ contain eigenvalues of $\hat{A}'(\lambda)$, while the strip $l + 2m - a_1 + 1 - 2m < \text{Im } \lambda < l + 2m - a_2 + 1 - 2m$ contains no eigenvalues of $\hat{A}'(\lambda)$. By assumption, the operator $A(\omega) : E_{a_0}^{l+2m}(\mathbb{R}^2) \rightarrow E_{a_0}^l(\mathbb{R}^2)$, $\omega \in S^{n-3}$, is an isomorphism. Therefore, by Lemmas 3.6 and 3.3, the operator $A(\omega) : E_{a_0-l}^{2m}(\mathbb{R}^2) \rightarrow E_{a_0-l}^0(\mathbb{R}^2)$ is also an isomorphism. Now it follows from Lemma 3.8 that the operator

$$A'(\omega) : E_{l+2m-a_0}^{2m}(\mathbb{R}^2) \rightarrow E_{l+2m-a_0}^0(\mathbb{R}^2)$$

is an isomorphism. Applying part 1 of this proof to the operator $A'(\omega)$, we conclude that $\dim \mathcal{N}(A'(\omega)) > 0$ for $A'(\omega) : E_b^{2m}(\mathbb{R}^2) \rightarrow E_b^0(\mathbb{R}^2)$, where $b > l + 2m - a_2$. Therefore, by virtue of Lemma 3.8, $d > 0$ for $A(\omega) : E_{a-l}^{2m}(\mathbb{R}^2) \rightarrow E_{a-l}^0(\mathbb{R}^2)$, where $2m - (a - l) > l + 2m - a_2$, i.e., $a < a_2$.

2b. It remains to prove that $\dim \mathcal{R}(A(\omega))^\perp = d$ for

$$A(\omega) : E_a^{l+2m}(\mathbb{R}^2) \rightarrow E_a^l(\mathbb{R}^2).$$

Due to part 1b of the proof, Eq. (3.32) with right-hand side $f_0 \in E_{a-l}^0(\mathbb{R}^2)$ has a solution $v \in E_{a-l}^{2m}(\mathbb{R}^2)$ iff

$$(f_0, f_j)_{E_{a-l}^0(\mathbb{R}^2)} = 0, \quad j = 1, \dots, d, \quad (3.42)$$

where $f_1, \dots, f_d \in E_{a-l}^0(\mathbb{R}^2)$ are linearly independent functions. It follows from Lemma 3.6 that conditions (3.42) are necessary and sufficient for Eq. (3.32) with right-hand side $f_0 \in E_a^l(\mathbb{R}^2)$ to have a solution $v \in E_a^{l+2m}(\mathbb{R}^2)$. It follows from the Schwarz inequality and from the boundedness of the embedding $E_a^l(\mathbb{R}^2) \subset E_{a-l}^0(\mathbb{R}^2)$ that

$$|(f_0, f_j)_{E_{a-l}^0(\mathbb{R}^2)}| \leq \|f_0\|_{E_{a-l}^0(\mathbb{R}^2)} \|f_j\|_{E_{a-l}^0(\mathbb{R}^2)} \leq c \|f_0\|_{E_a^l(\mathbb{R}^2)} \|f_j\|_{E_{a-l}^0(\mathbb{R}^2)},$$

where $c > 0$ does not depend on f_0 . Hence, by Riesz' theorem, there are functions $F_j \in E_a^l(\mathbb{R}^2)$, $j = 1, \dots, d$, such that

$$(f_0, f_j)_{E_{a-l}^0(\mathbb{R}^2)} = (f_0, F_j)_{E_a^l(\mathbb{R}^2)} \quad \text{for all} \quad f_0 \in E_a^l(\mathbb{R}^2),$$

and the functions F_j are linearly independent. Thus, $\dim \mathcal{R}(A(\omega))^\perp$ in $E_a^l(\mathbb{R}^2)$ is equal to d . \square

Proof of Lemma 3.10. 1. First, we show that the operator

$$A(\omega) : E_{l+m}^{l+2m}(\mathbb{R}^2) \rightarrow E_{l+m}^l(\mathbb{R}^2)$$

is not an isomorphism for any $l \geq 0$. To do so, we prove that $\lambda_0 = i(1-m)$ is an eigenvalue of $\hat{A}(\lambda)$. Consider a homogeneous polynomial $q(y)$ of order $m-1$ and write it in the polar coordinates, $q(y) = r^{m-1} \tilde{q}(\varphi)$, where $\tilde{q} \in C_{2\pi}^\infty[0, 2\pi]$. We have

$$0 = A(D_y, 0)q(y) = r^{-2m} \hat{A}(\varphi, D_\varphi, rD_r)(r^{m-1} \tilde{q}(\varphi)) = r^{-m-1} \hat{A}(\varphi, D_\varphi, i(1-m)) \tilde{q}(\varphi).$$

Hence, $\lambda_0 = i(1-m)$ is an eigenvalue and $\tilde{q}(\varphi)$ is the corresponding eigenvector. Since the line $\text{Im } \lambda = 1-m = m+1-2m$ contains the eigenvalue λ_0 , it follows from Lemma 3.3 that the operator $A(\omega) : E_{l+m}^{l+2m}(\mathbb{R}^2) \rightarrow E_{l+m}^l(\mathbb{R}^2)$ does not have the Fredholm property. Therefore, it is not an isomorphism.

2. Now we prove that the operator $A(\omega) : E_a^{2m}(\mathbb{R}^2) \rightarrow E_a^0(\mathbb{R}^2)$ is not an isomorphism for any a , $a \neq m$. Assume, to the contrary, that $A(\omega)$ is an isomorphism for some $a \neq m$. Then, by Lemma 3.8,

$$\text{the operator } A'(\omega) : E_{2m-a}^{2m}(\mathbb{R}^2) \rightarrow E_{2m-a}^0(\mathbb{R}^2) \text{ is an isomorphism.} \quad (3.43)$$

Note that

$$\overline{A(D_y, \omega)u(y)} \equiv [A'(D_{y'}, \omega)w(y')]_{|_{y'=-y}},$$

where $w(y') = \overline{u(-y')}$. It follows from this relation and from (3.43) that the operator $A(\omega) : E_{2m-a}^{2m}(\mathbb{R}^2) \rightarrow E_{2m-a}^0(\mathbb{R}^2)$ is also an isomorphism. This contradicts Lemma 3.12 because the strip bounded by the lines $\text{Im } \lambda = a+1-2m$ and $\text{Im } \lambda = (2m-a)+1-2m$ contains the eigenvalue $\lambda_0 = i(1-m)$ of the operator-valued function $\hat{A}(\lambda)$.

3. Finally, we prove that the operator $A(\omega) : E_a^{l+2m}(\mathbb{R}^2) \rightarrow E_a^l(\mathbb{R}^2)$ is not an isomorphism for all $\omega \in S^{n-3}$, $l > 0$, and $a \neq l+m$. Assume, to the contrary, that it is an isomorphism for some $\omega \in S^{n-3}$, $l > 0$, and $a \neq l+m$. Then, by Lemma 3.3, the line $\text{Im } \lambda = a+1-l-2m$ contains no eigenvalues of the operator-valued function $\hat{A}(\lambda)$. Therefore, according to part 2 of this proof, either $\dim \mathcal{N}(A(\omega)) > 0$ or $\dim \mathcal{R}(A(\omega))^\perp > 0$ for the operator $A(\omega) : E_{a-l}^{2m}(\mathbb{R}^2) \rightarrow E_{a-l}^0(\mathbb{R}^2)$.

Let $\dim \mathcal{N}(A(\omega)) > 0$ for the above operator $A(\omega)$. Hence, there exists a function $v \in E_{a-l}^{2m_l}(\mathbb{R}^2)$ such that $v \neq 0$ and $A(\omega)v = 0$. By Lemma 3.6, $v \in E_a^{l+2m}(\mathbb{R}^2)$, and hence $\dim \mathcal{N}(A(\omega)) > 0$ for $A(\omega) : E_a^{l+2m}(\mathbb{R}^2) \rightarrow E_a^l(\mathbb{R}^2)$. This contradicts our assumption.

Let $\dim \mathcal{R}(A(\omega))^\perp > 0$ for $A(\omega) : E_{a-l}^{2m}(\mathbb{R}^2) \rightarrow E_{a-l}^0(\mathbb{R}^2)$. Since $A(\omega) : E_a^{l+2m}(\mathbb{R}^2) \rightarrow E_a^l(\mathbb{R}^2)$ is an isomorphism, it follows that the equation

$$A(\omega)v = f_0$$

has a solution $v \in E_a^{l+2m}(\mathbb{R}^2) \subset E_{a-l}^{2m}(\mathbb{R}^2)$ for each $f_0 \in E_a^l(\mathbb{R}^2)$. On the other hand, $E_a^l(\mathbb{R}^2)$ is dense in $E_{a-l}^0(\mathbb{R}^2)$, while $\mathcal{R}(A(\omega))$ is closed in $E_{a-l}^0(\mathbb{R}^2)$ for $A(\omega) : E_{a-l}^{2m}(\mathbb{R}^2) \rightarrow E_{a-l}^0(\mathbb{R}^2)$. Therefore, $\mathcal{R}(A(\omega)) = E_{a-l}^0(\mathbb{R}^2)$, which contradicts the assumption that $\dim \mathcal{R}(A(\omega))^\perp > 0$. \square

Theorem 3.1 follows from Lemmas 3.4 and 3.10.

4 A Priori Estimates of Solutions in Bounded Domains

4.1

In this section, we obtain a priori estimates for solutions of nonlocal elliptic problems in weighted spaces. Combining these estimates with the existence of a right regularizer, which we construct in the next section, we prove the Fredholm property for the corresponding nonlocal operator. Let us discuss the choice of weighted spaces. For each set \mathcal{K}_j , $j = 1, 2, 3$, we may assume that either the set K in the definition of the spaces $H_a^k(Q) = H_a^k(Q, K)$ and $H_a^{k-1/2}(\Gamma) = H_a^{k-1/2}(\Gamma, K)$ contains the set \mathcal{K}_j or it does not. This is equivalent to whether or not right-hand sides and solutions of nonlocal problems in bounded domains have singularities near the set \mathcal{K}_j . If $\mathcal{K}_j \subset K$, then the model operators corresponding to the points of the set \mathcal{K}_j and playing a fundamental role in obtaining a priori estimates and constructing a right regularizer must be considered in weighted spaces; otherwise, in Sobolev spaces.

Consider the set $\mathcal{K}_1 = \partial Q \setminus \bigcup_i \Gamma_i$ of conjugation points. It is shown in [24, 29] (see also [10]) that generalized solutions of nonlocal problems can have power-law singularities near the set \mathcal{K}_1 . Therefore, we always assume that $\mathcal{K}_1 \subset K$, while the corresponding model operators act on weighted spaces in dihedral angles.

Consider the set $\mathcal{K}_3 \subset Q$. It follows from Theorem 3.1 that the model operator on weighted spaces in \mathbb{R}^n is not an isomorphism (moreover, one can show that it does not have even the Fredholm property on weighted spaces, cf. Remark 2.2 in [22, Chap. 6, Sec. 2]). Therefore, we assume that right-hand sides and solutions of nonlocal problems in bounded domains have no singularities inside the domain Q , while the corresponding model operator acts on Sobolev spaces. If $\mathcal{K}_3 = \emptyset$, this assumption leads to no difficulties. However, if $\mathcal{K}_3 \neq \emptyset$, the following difficulty arises. Take a point $g \in \overline{\Gamma_i} \cap \mathcal{K}$ such that $\omega_{is}(g) \in \mathcal{K}_3$, and let a function u belong to the weighted space H_a^{l+2m} near the point g and to the Sobolev space W^{l+2m} near the point $\omega_{is}(g)$. Since ω_{is} is a smooth nondegenerate transformation, it follows that the function $(B_{i\mu s}(x, D)u)(\omega_{is}(x))|_{\Gamma_i}$ occurring in nonlocal conditions (1.4) belongs to $W^{l+2m-m_{i\mu}-1/2}$ near the point g ; however, in general, it does not belong to $H_a^{l+2m-m_{i\mu}-1/2}$ near the point g . Therefore, the corresponding nonlocal operator appears to be unbounded on weighted spaces. To eliminate this obstacle, we additionally assume that $a > l + 2m - 1$ in the case $\mathcal{K}_3 \neq \emptyset$, which ensures the inclusion $W^{l+2m-m_{i\mu}-1/2} \subset H_a^{l+2m-m_{i\mu}-1/2}$ in a neighborhood of $g \in \overline{\Gamma_i} \cap \mathcal{K}$ (cf. Lemma 4.5 below).

Consider the set $\mathcal{K}_2 \subset \bigcup_i \Gamma_i$. We may either include or not include the set \mathcal{K}_2 in the set K . In the first case, we consider model local operators on weighted spaces. In the second case, we consider model local operators on Sobolev spaces. The advantage of the “weighted case” is that we solve a nonlocal problem in the whole scale of spaces (depending on the weight parameter $a \in \mathbb{R}$). However, the disadvantage is that we must impose some assumptions on the location of eigenvalues of an auxiliary problem with the parameter λ and require that an auxiliary operator with the parameter $\omega \in S^{n-3}$ be an isomorphism (the latter is often hard to verify), see Theorem 2.3. The advantage of the case of Sobolev spaces is that the model operators are isomorphisms without any additional assumptions. The disadvantage is that, if $\mathcal{K}_2 \neq \emptyset$, we must suppose $a > l + 2m - 1$ even if $\mathcal{K}_3 = \emptyset$ (the reason is similar to that in the above case $\mathcal{K}_3 \neq \emptyset$).

The following consistency condition integrates all the above cases.

Condition 4.1 (consistency condition). 1. If $\mathcal{K}_3 = \emptyset$, then either

- (a) $a \in \mathbb{R}$ and $K = \mathcal{K}_1 \cup \mathcal{K}_2$, or
- (b) $a > l + 2m - 1$ and $K = \mathcal{K}_1$.

2. If $\mathcal{K}_3 \neq \emptyset$, then $a > l + 2m - 1$ and either

- (a) $K = \mathcal{K}_1 \cup \mathcal{K}_2$, or
- (b) $K = \mathcal{K}_1$.

To conclude this subsection, we prove two auxiliary results. Denote

$$\mathcal{M}^\delta = \{x \in \mathbb{R}^n : \text{dist}(x, \mathcal{M}) < \delta\}$$

for any set $\mathcal{M} \subset \mathbb{R}^n$ and $\delta > 0$.

Lemma 4.1. Let $\zeta \in C^\infty(\mathbb{R}^n)$ be a function such that $\zeta(x) = 0$ for $x \in \mathcal{K}_1$. Then

$$\|\zeta v\|_{H_a^l(Q)} \leq c\delta \|v\|_{H_a^l(Q)} \quad (4.1)$$

for all $v \in H_a^l(Q)$ such that $\text{supp } v \subset \overline{Q} \cap \mathcal{K}_1^\delta$, where $c > 0$ does not depend on δ and v .

Proof. Since $\zeta(x) = 0$ for $x \in \mathcal{K}_1$, it follows from the Taylor formula that

$$|\zeta(x)| \leq k_1 \delta, \quad x \in \mathcal{K}_1^\delta. \quad (4.2)$$

Therefore, using (4.2), we obtain

$$\begin{aligned} \|\zeta v\|_{H_a^l(Q)}^2 &= \sum_{|\beta| \leq l} \int_{Q \cap \mathcal{K}_1^\delta} \rho^{2(a-l+|\beta|)} |\zeta D^\beta v|^2 dx \\ &\quad + \sum_{|\alpha|=1}^l \sum_{|\beta| \leq l-|\alpha|} \int_{Q \cap \mathcal{K}_1^\delta} \rho^{2|\alpha|} \rho^{2(a-l+|\beta|)} |D^\alpha \zeta D^\beta v|^2 dx \\ &\leq \sum_{|\beta| \leq l} k_1^2 \delta^2 \int_{Q \cap \mathcal{K}_1^\delta} \rho^{2(a-l+|\beta|)} |D^\beta v|^2 dx + \sum_{|\alpha|=1}^l \sum_{|\beta| \leq l-|\alpha|} k_2 \delta^{2|\alpha|} \int_{Q \cap \mathcal{K}_1^\delta} \rho^{2(a-l+|\beta|)} |D^\beta v|^2 dx, \end{aligned}$$

which implies (4.1). \square

Lemma 4.2. *Let $\zeta_\delta \in C^\infty(\mathbb{R}^n)$ be a family of functions such that $\text{supp } \zeta_\delta \subset \mathcal{K}_1^\delta$ and*

$$|D^\beta \zeta_\delta(x)| \leq c_1 \delta^{-|\beta|}, \quad x \in Q, \quad |\beta| \leq l, \quad (4.3)$$

where $c_1 > 0$ does not depend on δ . Then

$$\|\zeta_\delta u\|_{H_a^l(Q)} \leq c_2 \|u\|_{H_a^l(Q)} \quad (4.4)$$

for all $u \in H_a^l(Q)$, where $c_2 > 0$ does not depend on δ and u .

Proof. Using (4.3), we obtain

$$\begin{aligned} \|\zeta_\delta u\|_{H_a^l(Q)}^2 &= \sum_{|\alpha|+|\beta| \leq l} \int_{Q \cap \mathcal{K}_1^\delta} \rho^{2|\alpha|} \rho^{2(a-l+|\beta|)} |D^\alpha \zeta_\delta D^\beta u|^2 dx \\ &\leq k \sum_{|\beta| \leq l} \int_{Q \cap \mathcal{K}_1^\delta} \rho^{2(a-l+|\beta|)} |D^\beta u|^2 dx, \end{aligned}$$

which implies (4.4). \square

Remark 4.1. Lemmas 4.1 and 4.2 are true for the spaces $H_a^l(Q)$ replaced by $H_a^{l-1/2}(\Gamma)$, where Γ is a smooth $(n-1)$ -dimensional manifold such that $\bar{\Gamma} \subset \bar{Q}$ and $l-1/2 \geq 1/2$. To prove this, it suffices to use the corresponding bounded operator of extension acting from $H_a^{l-1/2}(\Gamma)$ to $H_a^l(Q)$.

4.2

We introduce the linear operator

$$\mathbf{L} = \{A, B_{i\mu}\}$$

corresponding to problem (1.3), (1.4). It follows from Lemma 4.6 (see below) that the operator $\mathbf{L} : H_a^{l+2m}(Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ is bounded.

The main result of this section is as follows.

Theorem 4.1. *Let Conditions 1.1–1.4 and 4.1 hold. Assume that the line $\text{Im } \lambda = a + 1 - l - 2m$ contains no eigenvalues of $\hat{\mathcal{L}}_g(\lambda)$ for any $g \in K$ and $\dim \mathcal{N}(\mathcal{L}_g(\omega)) = \text{codim } \mathcal{R}(\mathcal{L}_g(\omega)) = 0$ for any $g \in K$ and $\omega \in S^{n-3}$. Then the following estimate holds for all $u \in H_a^{l+2m}(Q)$:*

$$\|u\|_{H_a^{l+2m}(Q)} \leq c(\|\mathbf{L}u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \|u\|_{H_a^{l+2m-1}(Q)}), \quad (4.5)$$

where $c > 0$ does not depend on u .

Let A^0 and $B_{i\mu}^0$ denote the principal homogeneous parts of the operators $A(x, D)$ and $B_{i\mu s}(x, D)$, respectively. Set

$$B_{i\mu}^0 u = B_{i\mu 0}^0 u|_{\Gamma_i}.$$

For each $\varepsilon > 0$, we introduce a function $\xi = \xi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ such that $\xi(x) = 1$ for $x \in \mathcal{K}_1^{\varepsilon/2}$, $\text{supp } \xi(x) \subset \mathcal{K}_1^\varepsilon$, and

$$|D^\beta \xi(x)| \leq k_1 \varepsilon^{-|\beta|}, \quad x \in Q, \quad (4.6)$$

where $k_1 = k_1(\beta) > 0$ does not depend on ε . Since ω_{is} are C^∞ diffeomorphisms, it follows that

$$\text{supp } \xi(\omega_{is}(x)) \subset \mathcal{K}_1^{\varepsilon''}, \quad (4.7)$$

where $\varepsilon'' = \varepsilon''(i, s, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ ($i = 1, \dots, N_1$; $s = 0, \dots, S_i$).

We assume that $\varepsilon > 0$ is so small that

$$0 < \varepsilon'' < \text{dist}(\mathcal{K}_1, \mathcal{K}_2 \cup \mathcal{K}_3)/4. \quad (4.8)$$

Later on, we will make additional assumptions concerning ε (see the proofs of Lemmas 4.3 and 5.2).

Consider the operators

$$\begin{aligned} B_{i\mu}^1 u &= \sum_{s=1}^{S_i} (B_{i\mu s}^0(x, D)(\xi u))(\omega_{is}(x))|_{\Gamma_i}, \\ B_{i\mu}^2 u &= \sum_{s=1}^{S_i} (B_{i\mu s}^0(x, D)((1 - \xi)u))(\omega_{is}(x))|_{\Gamma_i}, \\ B_{i\mu}^3 &= B_{i\mu} - B_{i\mu}^1 - B_{i\mu}^2, \quad A^1 = A - A^0. \end{aligned}$$

The operators $B_{i\mu}^1$ correspond to nonlocal terms supported near the set \mathcal{K}_1 and the operators $B_{i\mu}^2$ to nonlocal terms supported outside the set \mathcal{K}_1 , while $B_{i\mu}^3$ and A^1 correspond to lower-order terms (compact perturbations).

Denote $B^k = \{B_{i\mu}^k\}_{i,\mu}$, $k = 0, \dots, 3$, $B = B^0 + \dots + B^3$, and $C = B^0 + B^1$.

Along with the operator $\mathbf{L} = (A, B)$ we consider the bounded operators

$$\mathbf{L}^0 = (A^0, B^0) : H_a^{l+2m}(Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma), \quad \mathbf{L}^1 = (A^0, C) : H_a^{l+2m}(Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma).$$

We first obtain an a priori estimate (similar to (4.5)) for the operator \mathbf{L}^1 with sufficiently small ε . Then we prove a fundamental property of the operators $B_{i\mu}^2$ related to the fact that the operators $B_{i\mu}^2$ correspond to nonlocal terms supported outside the set \mathcal{K}_1 . Combining these results will allow us to prove Theorem 4.1.

Lemma 4.3. *Let the conditions of Theorem 4.1 be fulfilled. Then there is an $\varepsilon > 0$ such that the following estimate holds for all $u \in H_a^{l+2m}(Q)$:*

$$\|u\|_{H_a^{l+2m}(Q)} \leq c(\|\mathbf{L}^1 u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \|u\|_{H_a^{l+2m-1}(Q)}), \quad (4.9)$$

where $c > 0$ does not depend on u .

Proof. 1. For any point $g \in \mathcal{K}_1$, denote by $\mathcal{O}(g)$ the orbit of g , see Sec. 1.1. By Condition 1.3, each orbit $\mathcal{O}(g)$ consists of finitely many points g_j , $j = 1, \dots, N(g)$. Set

$$\chi_m = \chi_m(g) = \min_{j, \rho, k, s} \chi_{j\rho ks} \quad (j, k = 1, \dots, N(g); \rho = 1, 2; s = 0, 1, \dots, S_{j\rho k}).$$

Clearly, $\chi_m \leq 1$. Let $x' \rightarrow x(g, j)$ be the change of variables inverse to the change of variables $x \rightarrow x'(g, j)$ from Sec. 1.1. The transformation $x' \rightarrow x(g, j)$ takes each ball $B_{\chi_m \delta}$ onto some neighborhood $\hat{B}_\delta(g_j)$ of the point g_j in such a way that the diameter of $\hat{B}_\delta(g_j)$ tends to zero as $\delta \rightarrow 0$. (Note that $\hat{B}_\delta(g_j)$ need not be a ball.) For each orbit $\mathcal{O}(g)$, we take a sufficiently small number $\delta = \delta(g) > 0$ such that $\hat{B}_\delta(g_j) \subset \hat{V}(g_j)$, $j = 1, \dots, N(g)$, and the operator \mathcal{L}_g'' has a bounded inverse for $\delta = \delta(g)$ (see Corollary 2.1).

It is clear that the union

$$\bigcup_{g \in \mathcal{K}_1} \bigcup_{j=1}^{N(g)} \hat{B}_{\delta(g)}(g_j)$$

covers the set \mathcal{K}_1 . We choose finitely many points $g^t \in \mathcal{K}_1$, $t = 1, \dots, T$, such that

$$\mathcal{K}_1 \subset \bigcup_{t,j} \hat{B}_{\delta(g^t)}(g_j^t).$$

Let functions $\varphi_j^t \in C_0^\infty(\mathbb{R}^n)$ form a partition of unity for the set \mathcal{K}_1 subordinated to the covering $\{\hat{B}_{\delta(g^t)}(g_j^t)\}$. Now we will pass from the partition of unity $\{\varphi_j^t\}$ to another partition of unity $\{\xi_j^t\}$ such that each function ξ_j^t , being written in the local coordinates $x' = (y', z')$, does not depend on y' in a neighborhood of the edge \mathcal{P} . To do so, we denote the function $\varphi_j^t(x)$ written in the variables $x' = (y', z')$ by $\varphi_j^t(y', z')$. Clearly, there is a number $a' < \chi_m \delta(g^t)$ such that

$$\varphi_j^t(0, z') = 0$$

for $a' < |z'| < \chi_m \delta(g^t)$. We can assume without loss of generality that the function $\varphi_j^t(0, z')$ is extended by zero for $|z'| \geq \chi_m \delta(g^t)$, and it remains to be infinitely differentiable.

Denote by $\psi^t \in C_0^\infty(\mathbb{R}^2)$ a function such that $\psi^t(y') = 1$ for $|y'| < \varepsilon'_1$ and $\psi_j^t(y') = 0$ for $|y'| > 2\varepsilon'_1$, where $\varepsilon'_1 > 0$ is so small that

$$\{(2\varepsilon'_1)^2 + (a')^2\}^{1/2} < \chi_m \delta(g^t), \quad (4.10)$$

and ε'_1 does not depend on ε .

Set

$$\xi_j^t(y', z') = \psi^t(y') \varphi_j^t(0, z').$$

By virtue of (4.10), we have

$$\text{supp } \xi_j^t(y', z') \subset B_{\chi_m \delta(g^t)}. \quad (4.11)$$

It is also clear that

$$\xi_j^t(y', z') = \varphi_j^t(0, z'), \quad |y'| < \varepsilon'_1. \quad (4.12)$$

Denote by $\xi_j^t(x)$ the functions $\xi_j^t(y', z')$ written in the variables $x = x(g^t, j)$. Since $\text{supp } \xi_j^t(x) \subset \hat{B}_{\delta(g^t)}(g_j^t)$, we can extend each function $\xi_j^t(x)$ by zero outside the neighborhood $\hat{B}_{\delta(g^t)}(g_j^t)$ to obtain the function infinitely differentiable on \mathbb{R}^n .

Obviously, we have

$$\sum_{j,t} \xi_j^t(x) = \sum_{j,t} \varphi_j^t(x) = 1, \quad x \in \mathcal{K}_1. \quad (4.13)$$

2. Take an arbitrary function $u \in H_a^{l+2m}(Q)$. If $g_j^t \in \overline{\Gamma_i}$ and $x \in \hat{V}(g_j^t)$, then it follows from Condition 1.3 that $\omega_{is}(x) \in V(g_p^t)$ for some $1 \leq p \leq N(g^t)$. Denote $u_p^t(x) = u(x)$ for $x \in Q \cap V(g_p^t)$. Then $u_p^t(\omega_{is}(x)) = u(\omega_{is}(x))$ for $x \in Q \cap \hat{V}(g_j^t)$. Let $x \rightarrow x'(g^t, j)$ be the change of variables from Sec. 1.1, corresponding to the orbit $\mathcal{O}(g^t)$. Denote the functions ξ_j^t and u_j^t written in the new variables x' by the same symbols (which leads to no confusion) and let $u^t = (u_1^t, \dots, u_{N(g^t)}^t)$. Applying Corollary 2.1, we obtain

$$\|\xi_q^t u\|_{H_a^{l+2m}(Q)} \leq k_1 \|\xi_q^t u_q^t\|_{H_a^{l+2m}(\Theta_q)} \leq k_1 \|\xi_q^t u^t\|_{\mathcal{H}_a^{l+2m}(\Theta)} \leq k_2 \|\mathcal{L}_{g^t}''(\xi_q^t u^t)\|_{\mathcal{H}_a^l(\Theta, \Gamma)},$$

where $q = 1, \dots, N(g^t)$, while $k_1, k_2, \dots > 0$ do not depend on u .

It follows from (4.11) that

$$\xi_q^t(\mathcal{G}_{j\rho ks} y', z') = 0, \quad |x'| > \delta(g^t).$$

Therefore, $\mathcal{L}_{g^t}''(\xi_q^t u^t) = \mathcal{L}_{g^t}'(\xi_q^t u^t)$, and, by using Leibniz' formula, we have

$$\begin{aligned} \|\xi_q^t u\|_{H_a^{l+2m}(Q)} &\leq k_2 \|\mathcal{L}_{g^t}'(\xi_q^t u^t)\|_{\mathcal{H}_a^l(\Theta, \Gamma)} \leq k_3 \left(\|\mathbf{L}^1 u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \|u\|_{H_a^{l+2m-1}(Q)} \right. \\ &\quad \left. + \sum_{h=1,2} \sum_{j,\rho,\mu} \sum_{(k,s) \neq (j,0)} \|\Psi_{j\rho\mu ks}^h\|_{H_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})} \right), \end{aligned} \quad (4.14)$$

where

$$\begin{aligned} \Psi_{j\rho\mu ks}^1 &= (B_{j\rho\mu ks}^0(x', D_{y'}, D_{z'})((1-\xi)\xi_q^t u_k^t))(\mathcal{G}_{j\rho ks} y', z')|_{\Gamma_{j\rho}}, \\ \Psi_{j\rho\mu ks}^2 &= (\xi_q^t(\mathcal{G}_{j\rho ks} y', z') - \xi_q^t(y', z'))(B_{j\rho\mu ks}^0(x', D_{y'}, D_{z'})((\xi u_k^t))(\mathcal{G}_{j\rho ks} y', z')|_{\Gamma_{j\rho}}. \end{aligned}$$

Denote by $\xi_{qk}^t(x)$ the function $\xi_q^t(y', z')$ written in the variables $x = x(g^t, k)$. Clearly, $\text{supp } \xi_{qk}^t \subset \hat{B}_{\delta(g^t)}(g_k^t)$. Passing to the variables $\hat{x} = \omega_{is}(x)$, we estimate the norm of $\Psi_{j\rho\mu ks}^1$ in the following way:

$$\begin{aligned} \|\Psi_{j\rho\mu ks}^1\|_{H_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})} &\leq k_4 \|B_{i\mu s}^0(\hat{x}, D)((1-\xi)\xi_{qk}^t u)(\hat{x})|_{\omega_{is}(\Gamma_i)}\|_{H_a^{l+2m-m_{i\mu}-1/2}(\omega_{is}(\Gamma_i))}. \end{aligned}$$

Denote

$$Q_b = \{x \in Q : \text{dist}(x, \partial Q) > b\},$$

where $b > 0$. Since $\hat{B}_{\delta(g^t)}(g_k^t) \subset \hat{V}(g_k^t)$, it follows from Condition 1.3 that the set

$$\Omega_0 = (\overline{\omega_{is}(\Gamma_i)} \cap \hat{B}_{\delta(g^t)}(g_k^t)) \setminus \mathcal{K}_1^{\varepsilon/2}$$

intersects neither \mathcal{K}_1 nor \mathcal{K}_2 . Therefore, there exists a number $b = b(\varepsilon) > 0$ such that $\Omega_0 \subset Q_b$. Since

$$\text{supp } ((1-\xi)\xi_{qk}^t)|_{\omega_{is}(\Gamma_i)} \subset \Omega_0 \subset Q_b,$$

using the last inequality, the equivalence of the norms in the spaces $H_a^{l+2m}(Q_b)$ and $W^{l+2m}(Q_b)$, and Lemma 3.5, we obtain

$$\begin{aligned} \|\Psi_{j\rho\mu ks}^1\|_{H_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})} &\leq k_5\|u\|_{H_a^{l+2m}(Q_b)} \leq k_6\|u\|_{W^{l+2m}(Q_b)} \\ &\leq k_7(\|A^0u\|_{W^l(Q_{b/2})} + \|u\|_{L_2(Q_{b/2})}) \leq k_8(\|A^0u\|_{H_a^l(Q)} + \|u\|_{H_a^{l+2m-1}(Q)}). \end{aligned} \quad (4.15)$$

Now let us estimate the norm of $\Psi_{j\rho\mu ks}^2$. By virtue of (4.12),

$$\xi_q^t(\mathcal{G}_{j\rho ks}y', z') - \xi_q^t(y', z') = 0, \quad |y'| < \varepsilon'_1 / \max(1, \chi_{j\rho ks}).$$

Therefore, similarly to (4.15), we obtain

$$\|\Psi_{j\rho\mu ks}^2\|_{H_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})} \leq k_9(\|A^0u\|_{H_a^l(Q)} + \|u\|_{H_a^{l+2m-1}(Q)}). \quad (4.16)$$

It follows from (4.14), (4.15), and (4.16) that

$$\|\xi_q^t u\|_{H_a^{l+2m}(Q)} \leq k_{10}(\|\mathbf{L}^1 u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \|u\|_{H_a^{l+2m-1}(Q)}). \quad (4.17)$$

Setting

$$\xi_0(x) = \sum_{\ell, q} \xi_q^\ell(x) \quad (4.18)$$

and using inequality (4.17), we have

$$\|\xi_0 u\|_{H_a^{l+2m}(Q)} \leq k_{11}(\|\mathbf{L}^1 u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \|u\|_{H_a^{l+2m-1}(Q)}) \quad (4.19)$$

(note that k_{11} depends on ε).

3. Using a partition of unity, Theorem 2.3, Leibniz' formula, and a priori estimates of solutions for elliptic problems in the interior of Q and near a smooth part of the boundary, we obtain

$$\begin{aligned} \|(1 - \xi_0)u\|_{H_a^{l+2m}(Q)} &\leq c_1(\|(1 - \xi_0)\mathbf{L}^0 u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \|u\|_{H_a^{l+2m-1}(Q)}) \\ &\leq c_2\left(\|(1 - \xi_0)\mathbf{L}^1 u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \sum_{i, \mu} \|(1 - \xi_0)B_{i\mu}^1 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} + \|u\|_{H_a^{l+2m-1}(Q)}\right), \end{aligned} \quad (4.20)$$

where $c_1, c_2, \dots > 0$ do not depend on u and ε (we recall that the function ξ_0 does not depend on ε).

It follows from (4.13) and (4.18) that $1 - \xi_0(x) = 0$ for $x \in \mathcal{K}_1$. On the other hand, $\text{supp } \xi(\omega_{is}(x)) \subset \mathcal{K}_1^{\varepsilon''}$ due to (4.7). Therefore, applying Lemma 4.1 and Remark 4.1 and taking into account that ω_{is} are C^∞ diffeomorphisms, we have

$$\|(1 - \xi_0)B_{i\mu}^1 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq c_3\varepsilon''\|B_{i\mu}^1 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq c_4\varepsilon''\|\xi u\|_{H_a^{l+2m}(Q)}.$$

Further, using the relation $\text{supp } \xi \subset \mathcal{K}_1^\varepsilon$, inequalities (4.6), and Lemma 4.2, we obtain from the last estimate that

$$\|(1 - \xi_0)B_{i\mu}^1 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq c_5\varepsilon''\|u\|_{H_a^{l+2m}(Q)}.$$

Combining this estimate with (4.20) yields

$$\|(1 - \xi_0)u\|_{H_a^{l+2m}(Q)} \leq c_6(\|\mathbf{L}^1 u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \varepsilon''\|u\|_{H_a^{l+2m}(Q)} + \|u\|_{H_a^{l+2m-1}(Q)}). \quad (4.21)$$

Choosing ε in the definition of the function ξ so small that $c_6\varepsilon'' = 1/2$ and using inequalities (4.19) and (4.21), we complete the proof. \square

4.3

In this subsection, we prove Theorem 4.1. First, we formulate some results on properties of weighted spaces, which are needed below.

Lemma 4.4. *Let $Q_1 \subset \mathbb{R}^n$ be a bounded domain such that $\overline{Q} \subset Q_1$. Assume that the set K_1 in the definition of the space $H_a^k(Q_1) = H_a^k(Q_1, K_1)$ coincides with the set K in the definition of the space $H_a^k(Q) = H_a^k(Q, K)$. Then, for any function $v \in H_a^k(Q)$, there exists a function $v_1 \in H_a^k(Q_1)$ such that $v_1(x) = v(x)$ for $x \in Q$ and*

$$\|v_1\|_{H_a^k(Q_1)} \leq c\|v\|_{H_a^k(Q)},$$

where $c > 0$ does not depend on v .

Lemma 4.4 is proved in [27, Sec. 3].

Lemma 4.5. *Let $a > l + 2m - 1$. Assume that $\delta > 0$ satisfies the following conditions:*

1. $\mathcal{K}_1^\delta \cap (\mathcal{K}_2 \cup \mathcal{K}_3) = \emptyset$ if $\mathcal{K}_2 \cup \mathcal{K}_3 \neq \emptyset$,
2. $\mathcal{K}_2^\delta \cap (\mathcal{K} \setminus \mathcal{K}_2) = \emptyset$ if $\mathcal{K}_2 \neq \emptyset$,
3. $\mathcal{K}_3^\delta \subset Q$ and $\mathcal{K}_3^\delta \cap (\mathcal{K} \setminus \mathcal{K}_3) = \emptyset$ if $\mathcal{K}_3 \neq \emptyset$,
4. $\delta > 0$ is arbitrary if $\mathcal{K}_2 \cup \mathcal{K}_3 = \emptyset$.

Then

$$\|u\|_{H_a^{l+2m}(\mathcal{K}_j^\delta \cap Q)} \leq c_1 \|u\|_{W^{l+2m}(\mathcal{K}_j^\delta \cap Q)} \quad (4.22)$$

for all $u \in W^{l+2m}(\mathcal{K}_j^\delta \cap Q)$ if $\mathcal{K}_j \neq \emptyset$ ($j = 1, 2$) and

$$\|u\|_{H_a^{l+2m}(\mathcal{K}_3^\delta)} \leq c_2 \|u\|_{W^{l+2m}(\mathcal{K}_3^\delta)} \quad (4.23)$$

for all $u \in W^{l+2m}(\mathcal{K}_3^\delta)$ if $\mathcal{K}_3 \neq \emptyset$, where $c_1, c_2 > 0$ do not depend on u .

Lemma 4.5 is proved in [27] (see also Lemma 5.2 in [17]).

The following result is also obtained in [27, Sec. 3]. It means that the operators $B_{i\mu}^2$ correspond to nonlocal terms supported outside the set \mathcal{K}_1 . For the reader's convenience, we give the proof of this result.

Lemma 4.6. *Let Condition 4.1 hold. Then there exists a number $\varkappa = \varkappa(\varepsilon) > 0$ such that*

$$\|B_{i\mu}^2 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq c_1 \|u\|_{H_a^{l+2m}(Q \setminus \overline{\mathcal{K}_1^{2\varkappa}})} \quad (4.24)$$

for all $u \in H_a^{l+2m}(Q \setminus \overline{\mathcal{K}_1^{2\varkappa}})$; furthermore, there exists a number $\sigma = \sigma(\varkappa)$ such that

$$\|B_{i\mu}^2 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i \setminus \overline{\mathcal{K}_1^{2\varkappa}})} \leq c_2 \|u\|_{H_a^{l+2m}(Q_\sigma)} \quad (4.25)$$

for all $u \in H_a^{l+2m}(Q_\sigma)$; here $i = 1, \dots, N_0$; $\mu = 1, \dots, m$; $c_1, c_2 > 0$ do not depend on u .

Proof. 1. It suffices to show that inequalities (4.24) and (4.25) are valid for the function

$$\varphi_{i\mu s} = (B_{i\mu s}^0(x, D)((1 - \xi)u))(\omega_{is}(x))|_{\Gamma_i}$$

substituted for $B_{i\mu}^2 u$.

2. Let $\overline{\omega_{is}(\Gamma_i)} \cap \mathcal{K}_2 \neq \emptyset$. We assume without loss of generality that $\overline{\omega_{is}(\Omega_i)} \cap \mathcal{K}_2 = \overline{\omega_{is}(\Gamma_i)} \cap \mathcal{K}_2$, see Fig. 4.1. Let U be an extension of the function $(1 - \xi)u$ to $Q \cup \omega_{is}(\Omega_i)$, defined by Lemma 4.4

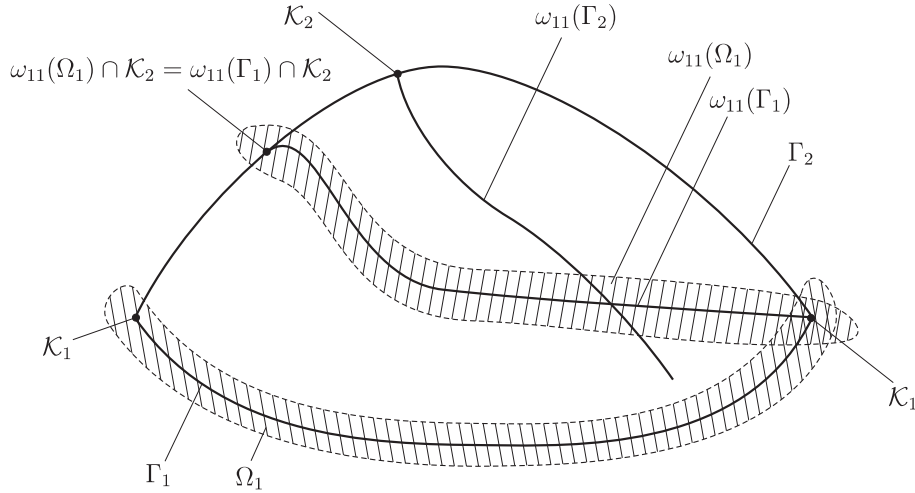


Figure 4.1: The domain Q

and satisfying the inequality

$$\|U\|_{H_a^{l+2m}(Q \cup \omega_{is}(\Omega_i))} \leq k_1 \|(1 - \xi)u\|_{H_a^{l+2m}(Q)}, \quad (4.26)$$

where $k_1, k_2, \dots > 0$ do not depend on u .

Set

$$\Phi_{i\mu s}(x) = (B_{i\mu s}^0(x, D)U)(\omega_{is}(x)), \quad x \in \Omega_i.$$

Clearly,

$$\varphi_{i\mu s} = \Phi_{i\mu s}|_{\Gamma_i}.$$

Introducing the new variable $\hat{x} = \omega_{is}(x)$, applying Lemma 4.5 if $\mathcal{K}_j \neq \emptyset$ and $\mathcal{K}_j \not\subset K$, $j = 2, 3$, and using inequality (4.26), we obtain

$$\begin{aligned} \|\varphi_{i\mu s}\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} &\leq \|\Phi_{i\mu s}\|_{H_a^{l+2m-m_{i\mu}}(\Omega_i)} \\ &\leq k_2 \|B_{i\mu s}^0(\hat{x}, D)U(\hat{x})\|_{H_a^{l+2m-m_{i\mu}}(\omega_{is}(\Omega_i))} \leq k_3 \|(1-\xi)u\|_{H_a^{l+2m}(Q)}. \end{aligned} \quad (4.27)$$

Thus, setting $2\kappa = \varepsilon/2$, we see that (4.27) implies (4.24).

Since the transformation ω_{is} is continuous and $\omega_{is}(\Gamma_i) \subset Q$, it follows that $\omega_{is}(\Gamma_i \setminus \overline{\mathcal{K}_1^\kappa}) \subset Q_{2\sigma}$ for sufficiently small $\sigma > 0$. Introduce a function $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ for $x \in Q_{2\sigma}$ and $\eta(x) = 0$ for $x \notin Q_\sigma$.

Suppose that the function ηu is extended by zero outside Q . Set

$$\Psi_{i\mu s}(x) = (B_{i\mu s}^0(x, D)(\eta(1-\xi)u))(\omega_{is}(x)), \quad x \in \Omega_i.$$

It is clear that

$$\varphi_{i\mu s}|_{\Gamma_i \setminus \overline{\mathcal{K}_1^\kappa}} = \Psi|_{\Gamma_i \setminus \overline{\mathcal{K}_1^\kappa}}.$$

Hence, applying Lemma 4.5 if $\mathcal{K}_j \neq \emptyset$ and $\mathcal{K}_j \not\subset K$, $j = 2, 3$, we obtain

$$\begin{aligned} \|\varphi_{i\mu s}\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i \setminus \overline{\mathcal{K}_1^\kappa})} &\leq \|\Psi_{i\mu s}\|_{H_a^{l+2m-m_{i\mu}}(\Omega_i)} \\ &\leq k_4 \|B_{i\mu s}^0(\hat{x}, D)(\eta(1-\xi)u)(\hat{x})\|_{H_a^{l+2m-m_{i\mu}}(\omega_{is}(\Omega_i))} \leq k_5 \|u\|_{H_a^{l+2m}(Q_\sigma)}. \end{aligned}$$

3. If $\overline{\omega_{is}(\Gamma_i)} \cap \mathcal{K}_2 = \emptyset$, then $\overline{\omega_{is}(\Gamma_i)} \setminus \mathcal{K}_1^{\varepsilon/2} \subset Q$. Therefore, similarly to the above, we obtain

$$\|\varphi_{i\mu s}\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq k_5 \|u\|_{H_a^{l+2m}(Q_\sigma)}.$$

This proves inequalities (4.24) and (4.25). \square

Proof of Theorem 4.1. 1. Take an arbitrary function $u \in H_a^{l+2m}(Q)$. It follows from Lemma 4.3 that

$$\|u\|_{H_a^{l+2m}(Q)} \leq k_1 \left(\|\mathbf{L}u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \|A^1 u\|_{H_a^l(Q)} + \sum_{i, \mu} \sum_{k=2,3} \|B_{i\mu}^k u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \right), \quad (4.28)$$

where $k_1, k_2, \dots > 0$ do not depend on u .

It follows from the boundedness of the domain Q and from Lemma 4.5 that

$$\|A^1 u\|_{H_a^l(Q)} + \sum_{i, \mu} \|B_{i\mu}^3 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq k_2 \|u\|_{H_a^{l+2m-1}(Q)}. \quad (4.29)$$

2. Consider a function $\eta \in C^\infty(\mathbb{R}^n)$ such that

$$\eta(x) = 1 \quad \text{for } x \in \mathbb{R}^n \setminus \overline{\mathcal{K}_1^{2\kappa}}, \quad \eta(x) = 0 \quad \text{for } x \in \mathcal{K}_1^\kappa,$$

where $\kappa > 0$ is the constant occurring in Lemma 4.6.

It follows from inequality (4.24), from Lemma 4.3, and from Leibniz' formula that

$$\begin{aligned} \|B_{i\mu}^2 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} &\leq k_3 \|\eta u\|_{H_a^{l+2m}(Q)} \\ &\leq k_4 \left(\|\eta \mathbf{L}^1 u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \|u\|_{H_a^{l+2m-1}(Q)} + \sum_{i, \mu} \sum_{s \neq 0} \|\Psi_{i\mu s}\|_{H_a^{l+2m-m_{i\mu}-1/2}(\omega_{is}(\Gamma_i))} \right), \end{aligned} \quad (4.30)$$

where

$$\Psi_{i\mu s} = (\eta(x) - \eta(\omega_{is}^{-1}(x)))(B_{i\mu s}^0(x, D)(\xi u))(x)|_{\omega_{is}(\Gamma_i)}.$$

It is clear that, if $\overline{\omega_{is}(\Gamma_i)} \cap \mathcal{K}_1^\varepsilon = \emptyset$, then $\Psi_{i\mu s} = 0$. Let $\overline{\omega_{is}(\Gamma_i)} \cap \mathcal{K}_1^\varepsilon \neq \emptyset$. We claim that

$$\text{supp } \Psi_{i\mu s} \subset Q_b \quad (4.31)$$

for some $b > 0$. Indeed,

$$\omega_{is}(\Gamma_i) \subset Q \quad \text{and} \quad \text{supp } \xi \subset \overline{\mathcal{K}_1^\varepsilon}. \quad (4.32)$$

Therefore, by virtue of (4.8), it suffices to show that $\Psi_{i\mu s}(x) = 0$ for x in some neighborhood of \mathcal{K}_1 . Since

$$\eta(x) = 0, \quad x \in \overline{\omega_{is}(\Gamma_i)} \cap \mathcal{K}_1^\kappa, \quad (4.33)$$

it remains to prove that

$$\eta(\omega_{is}^{-1}(x)) = 0, \quad x \in \overline{\omega_{is}(\Gamma_i)} \cap \mathcal{K}_1^d, \quad (4.34)$$

for a sufficiently small $d > 0$. Note that, if $\overline{\omega_{is}(\Gamma_i)} \cap \mathcal{K}_1 = \emptyset$, then (4.31) follows from (4.8) and (4.32). If $\overline{\omega_{is}(\Gamma_i)} \cap \mathcal{K}_1 \neq \emptyset$ for some i and s , then $\mathcal{K}_{1\nu} \subset \overline{\omega_{is}(\Gamma_i)}$ for some ν and $\omega_{is}^{-1}(\mathcal{K}_{1\nu}) \subset \mathcal{K}_1$. Hence, there exists a sufficiently small $d > 0$ such that $\mathcal{K}_{1\nu}^d \subset \omega_{is}(\Omega_i)$ and $\omega_{is}^{-1}(\mathcal{K}_{1\nu}^d) \subset \mathcal{K}_1^\kappa$ (because the transformations ω_{is}^{-1} are smooth). Clearly, (4.34) holds in this case. Thus, we obtain (4.31).

It follows from (4.31) and Lemma 3.5 that

$$\|\Psi_{i\mu s}\|_{H_a^{l+2m-m_{i\mu}-1/2}(\omega_{is}(\Gamma_i))} \leq k_5(\|Au\|_{H_a^l(Q)} + \|u\|_{H_a^{l+2m-1}(Q)}).$$

Using this inequality, we infer from (4.30) that

$$\begin{aligned} \|B_{i\mu}^2 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} &\leq k_6 \left(\|Lu\|_{\mathcal{H}_a^l(Q, \Gamma)} + \|u\|_{H_a^{l+2m-1}(Q)} + \sum_{i, \mu} \|\eta B_{i\mu}^2 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \right). \end{aligned} \quad (4.35)$$

By virtue of (4.25), Lemma 3.5, and Leibniz' formula, we have

$$\|\eta B_{i\mu}^2 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq k_7 \|u\|_{H_a^{l+2m}(Q_\sigma)} \leq k_8 (\|Au\|_{H_a^l(Q)} + \|u\|_{H_a^{l+2m-1}(Q)}). \quad (4.36)$$

Combining estimates (4.28), (4.29), (4.35), and (4.36), we obtain the desired estimate (4.5). \square

5 The Fredholm Property of Nonlocal Elliptic Problems

5.1

In this section, we prove the main result of the paper concerning the Fredholm property of nonlocal elliptic problems in weighted spaces. This result can be formulated as follows.

Theorem 5.1. *Let Conditions 1.1–1.4 and 4.1 hold. Assume that the line $\text{Im } \lambda = a + 1 - l - 2m$ contains no eigenvalues of $\hat{\mathcal{L}}_g(\lambda)$ for any $g \in K$ and $\dim \mathcal{N}(\mathcal{L}_g(\omega)) = \text{codim } \mathcal{R}(\mathcal{L}_g(\omega)) = 0$ for any $g \in K$ and $\omega \in S^{n-3}$. Then the operator $\mathbf{L} : H_a^{l+2m}(Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ has the Fredholm property.*

Due to Theorem 16.4 in [18] (about compact perturbations of Fredholm operators), it suffices to prove Theorem 5.1 and the other assertions of this section for $A^1 = 0$ and $B^3 = 0$. Therefore, we assume that $A^1 = 0$ and $B^3 = 0$ throughout this section.

Corollary 5.1. *Let the conditions of Theorem 5.1 be fulfilled. Then $\text{ind } \mathbf{L} = \text{ind } \mathbf{L}^1$.*

Proof. We introduce the operator

$$L_t u = \{A^0 u, Cu + (1-t)B^2 u\}.$$

We have $L_0 = \mathbf{L}$ (because $A^1 = 0$ and $B^3 = 0$) and $L_1 = \mathbf{L}^1$.

By Theorem 5.1, the operators L_t have the Fredholm property for all t . Furthermore, for any t_0 and t , the following estimate holds:

$$\|L_t u - L_{t_0} u\|_{\mathcal{H}_a^l(Q, \Gamma)} \leq k_{t_0} |t - t_0| \cdot \|u\|_{H_a^{l+2m}(Q)},$$

where $k_{t_0} > 0$ does not depend on t . Therefore, by Theorem 16.2 in [18], we have $\text{ind } L_t = \text{ind } L_{t_0}$ for any t in a sufficiently small neighborhood of the point t_0 . Since t_0 is arbitrary, these neighborhoods cover the segment $[0, 1]$. Choosing a finite subcovering, we obtain the relations $\text{ind } \mathbf{L} = \text{ind } L_0 = \text{ind } L_1 = \text{ind } \mathbf{L}^1$. \square

The proof of Theorem 5.1 is based on the existence of a right regularizer for the operator \mathbf{L} .

Theorem 5.2. *Let the conditions of Theorem 5.1 be fulfilled. Then there exists a linear bounded operator $\mathbf{R} : \mathcal{H}_a^l(Q, \Gamma) \rightarrow H_a^{l+2m}(Q)$ such that*

$$\mathbf{L}\mathbf{R} = \mathbf{I} + \mathbf{T}$$

where \mathbf{I} and \mathbf{T} are the identity operator and a compact operator on $\mathcal{H}_a^l(Q, \Gamma)$, respectively.

Proof of Theorem 5.1. Assume that Theorem 5.2 is true. By Lemma 3.5 in [15], the embedding of $H_a^{l+2m}(Q)$ into $H_a^{l+2m-1}(Q)$ is compact. Therefore, by Theorem 7.1 in [18] and Theorem 4.1, $\dim \mathcal{N}(\mathbf{L}) < \infty$ and the range $\mathcal{R}(\mathbf{L})$ is closed in $\mathcal{H}_a^l(Q, \Gamma)$. On the other hand, Theorem 15.2 in [18] and Theorem 5.2 imply that $\text{codim } \mathcal{R}(\mathbf{L}) < \infty$. \square

Thus, it remains to prove Theorem 5.2.

5.2

First, we prove the following auxiliary result.

Lemma 5.1. *Let H be a Hilbert space and I the identity operator on H . Let M_ε and S_ε , $\varepsilon > 0$, be families of bounded operators on H such that*

$$\|M_\varepsilon\| \leq c_1\varepsilon, \quad \|S_\varepsilon\| \leq c_2, \quad (5.1)$$

where $c_1, c_2 > 0$ do not depend on ε , and the operators S_ε^2 are compact. Then the operators

$$L_\varepsilon = I + M_\varepsilon + S_\varepsilon$$

have the Fredholm property for sufficiently small $\varepsilon > 0$.

Proof. To prove the lemma, we will construct a right and a left regularizers for L_ε . We have

$$L_\varepsilon(I - (M_\varepsilon + S_\varepsilon)) = I - M_\varepsilon^2 - M_\varepsilon S_\varepsilon - S_\varepsilon M_\varepsilon - S_\varepsilon^2.$$

It follows from (5.1) that

$$\|M_\varepsilon^2 + M_\varepsilon S_\varepsilon + S_\varepsilon M_\varepsilon\| \leq c_3\varepsilon,$$

where $c_3 > 0$ does not depend on ε . Therefore, the operators $I - M_\varepsilon^2 - M_\varepsilon S_\varepsilon - S_\varepsilon M_\varepsilon$ have the Fredholm property by Theorem 16.2 in [18], provided that $\varepsilon > 0$ is sufficiently small. Further, using the fact that the operators S_ε^2 are compact and applying Theorem 16.4 in [18], we see that the operators $L_\varepsilon(I - (M_\varepsilon + S_\varepsilon))$ also have the Fredholm property. Now it follows from Theorem 15.2 in [18] that there exist bounded operators $R_{1\varepsilon}$ and compact operators $T_{1\varepsilon}$ such that

$$L_\varepsilon(I - (M_\varepsilon + S_\varepsilon))R_{1\varepsilon} = I + T_{1\varepsilon}. \quad (5.2)$$

Similarly, one can prove that there exist bounded operators $R_{2\varepsilon}$ and compact operators $T_{2\varepsilon}$ such that

$$R_{2\varepsilon}(I - (M_\varepsilon + S_\varepsilon))L_\varepsilon = I + T_{2\varepsilon}. \quad (5.3)$$

The conclusion of the lemma follows from relations (5.2) and (5.3) and from Theorems 15.2 and 14.3 in [18]. \square

To prove Theorem 5.2, we preliminarily consider the operator \mathbf{L}^1 , i.e., assume that nonlocal terms are supported near the set \mathcal{K}_1 .

Lemma 5.2. *Let the conditions of Theorem 5.1 be fulfilled and the number ε be sufficiently small. Then there exist a linear bounded operator $\mathbf{R}_1 : \mathcal{H}_a^l(Q, \Gamma) \rightarrow H_a^{l+2m}(Q)$ and a compact operator $\mathbf{T}_1 : \mathcal{H}_a^l(Q, \Gamma) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ such that*

$$\mathbf{L}^1 \mathbf{R}_1 = \mathbf{I} + \mathbf{T}_1.$$

Proof. 1. To construct a right regularizer, we consider a partition of unity $\{\xi_j^t\}$ different from that in the proof of Lemma 4.3.

For each orbit $\mathcal{O}(g)$, $g \in \mathcal{K}_1$, we denote by $\hat{B}_\delta(g_j)$ the same neighborhoods as in the proof of Lemma 4.3, and let $\{\varphi_j^t\}$ be the same partition of unity for \mathcal{K}_1 . We denote the function $\varphi_j^t(x)$ written in the variables $x' = (y', z')$ by $\varphi_j^t(y', z')$. Clearly, there is a number $a' < \chi_m \delta(g^t)$ such that

$$\varphi_j^t(0, z') = 0$$

for $a' < |z'| < \chi_m \delta(g^t)$. As before, we assume that $\varphi_j^t(0, z')$ is extended by zero for $|z'| \geq \chi_m \delta(g^t)$, and it remains to be infinitely differentiable.

Let $\hat{\varphi}^t \in C_0^\infty(\mathbb{R}^{n-2})$ be a function such that

$$\hat{\varphi}^t(z') = 1, \quad |z'| < a',$$

$$\hat{\varphi}^t(z') = 0, \quad |z'| > \hat{a}',$$

where $a' < \hat{a}' < \chi_m \delta(g^t)$.

Denote by $\psi^t, \hat{\psi}^t \in C_0^\infty(\mathbb{R}^2)$ functions such that $\psi^t(y') = 1$ for $|y'| \leq \varepsilon'_1$ and $\psi^t(y') = 0$ for $|y'| \geq 3\varepsilon'_1/2$, $\hat{\psi}^t(y') = 1$ for $|y'| \leq 3\varepsilon'_1/2$, and $\hat{\psi}^t(y') = 0$ for $|y'| \geq 2\varepsilon'_1$, where $\varepsilon'_1 > 0$ is so small that

$$\{(2\varepsilon'_1)^2 + (\hat{a}')^2\}^{1/2} < \chi_m \delta(g^t), \quad (5.4)$$

and ε'_1 does not depend on ε .

Set

$$\xi_j^t(y', z') = \psi^t(y') \varphi_j^t(0, z'), \quad \hat{\xi}^t(y', z') = \hat{\psi}^t(y') \hat{\varphi}^t(z').$$

By virtue of (5.4), we have

$$\text{supp } \xi_j^t(y', z') \subset B_{\chi_m \delta(g^t)}, \quad \text{supp } \hat{\xi}^t(y', z') \subset B_{\chi_m \delta(g^t)}. \quad (5.5)$$

It is also clear that

$$\xi_j^t(y', z') = \varphi_j^t(0, z'), \quad \hat{\xi}^t(y', z') = \hat{\varphi}^t(z'), \quad |y'| < \varepsilon'_1, \quad (5.6)$$

$$\hat{\xi}^t(y', z') \xi_j^t(y', z') = \xi_j^t(y', z'), \quad (y', z') \in \mathbb{R}^n. \quad (5.7)$$

Denote by $\xi_j^t(x)$ and $\hat{\xi}^t(x)$ the functions $\xi_j^t(y', z')$ and $\hat{\xi}^t(y', z')$, respectively, written in the variables $x = (g^t, j)$. Since $\text{supp } \xi_j^t(x) \subset \hat{B}_{\delta(g^t)}(g_j^t)$ and $\text{supp } \hat{\xi}^t(x) \subset \hat{B}_{\delta(g^t)}(g_j^t)$, we can extend these functions by zero outside the neighborhood $\hat{B}_{\delta(g^t)}(g_j^t)$ to obtain the functions infinitely differentiable on \mathbb{R}^n .

Obviously, we have

$$\sum_{j,t} \xi_j^t(x) = \sum_{j,t} \varphi_j^t(x) = 1, \quad x \in \mathcal{K}_1. \quad (5.8)$$

2. We set

$$M_H^t = \left\{ u \in H_a^{l+2m}(Q) : \text{supp } u \subset \bigcup_{j=1}^{N(g^t)} V(g_j^t) \right\},$$

$$\mathcal{M}_H^t = \{ v \in \mathcal{H}_a^{l+2m}(\Theta) : \text{supp } v \subset V(0) \}.$$

For all $u \in M_H^t$, denote $u_j^t(x) = u(x)$, $x \in V(g_j^t)$. We define the isomorphism $U^t : M_H^t \rightarrow \mathcal{M}_H^t$ by the formulas

$$(U^t u)_j(x') = u_j^t(x(x')), \quad x' \in \Theta_j \cap V(0); \quad (U^t u)_j(x') = 0, \quad x' \in \Theta_j \setminus V(0);$$

$j = 1, \dots, N(g^t)$.

We set

$$G_H^t = \left\{ f \in \mathcal{H}_a^l(Q, \Gamma) : \text{supp } f \subset \bigcup_{j=1}^{N(g^t)} V(g_j^t) \right\},$$

$$\mathcal{G}_H^t = \{ \Phi \in \mathcal{H}_a^l(\Theta, \Gamma) : \text{supp } \Phi \subset V(0) \}.$$

If $f = \{f_0, f_{i\mu}\} \in G_H^t$, then we denote by $f_j^t(x)$ and $f_{j\rho\mu}^t(x)$ the functions $f_0(x)$ and $f_{i\mu}(x)$ for $x \in V(g_j^t)$ and $x \in \Gamma_i \cap V(g_j^t)$, respectively, where ρ and j are such that the transformation $x \mapsto x'(g^t, j)$ maps $\Gamma_i \cap V(g_j^t)$ onto $\Gamma_{j\rho} \cap V(0)$. Further, we define the isomorphism $F^t : G_H^t \rightarrow \mathcal{G}_H^t$ by the formula

$$(F^t f)(x') = \{(F^t f)_j(x'), (F^t f)_{j\rho\mu}(x')\}.$$

Here

$$(F^t f)_j(x') = f_j^t(x(x')), \quad x' \in \Theta_j \cap V(0); \quad (F^t f)_j(x') = 0, \quad x' \in \Theta_j \setminus V(0);$$

$$(F^t f)_{j\rho\mu}(x') = f_{j\rho\mu}^t(x(x')), \quad x' \in \Gamma_{j\rho} \cap V(0); \quad (F^t f)_{j\rho\mu}(x') = 0, \quad x' \in \Gamma_{j\rho} \setminus V(0);$$

$j = 1, \dots, N(g^t)$; $\rho = 1, 2$; $\mu = 1, \dots, m$.

3. We set

$$R_{\mathcal{K}_1} f = \sum_t (U^t)^{-1} \left(\hat{\xi}^t (\mathcal{L}_{g^t}'')^{-1} F^t \left(\sum_q \xi_q^t f \right) \right). \quad (5.9)$$

Since the functions $\hat{\xi}^t$ and ξ_q^t do not depend on ε , it follows from Corollary 2.1 that

$$\|R_{\mathcal{K}_1} f\|_{H_a^{l+2m}(Q)} \leq c_1 \|f\|_{\mathcal{H}_a^l(Q, \Gamma)}, \quad (5.10)$$

where $c_1, c_2, \dots > 0$ depend neither on ε nor on a function occurring in the right-hand side.

Clearly, we have

$$\mathbf{L}^1 R_{\mathcal{K}_1} f = \mathbf{L} R_{\mathcal{K}_1} f + T_1 f, \quad (5.11)$$

where $T_1 : \mathcal{H}_a^l(Q, \Gamma) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ is a linear bounded operator given by

$$T_1 f = \left\{ 0, - \sum_{s=1}^{S_i} (B_{i\mu s}^0(x, D)((1-\xi)R_{\mathcal{K}_1} f))(\omega_{is}(x))|_{\Gamma_i} \right\}$$

(recall that $A^1 = 0$ and $B^3 = 0$). Moreover, using Lemmas 4.6 and 4.2 and estimate (5.10), we have

$$\|T_1 f\|_{\mathcal{H}_a^l(Q, \Gamma)} \leq c_2(\|R_{\mathcal{K}_1} f\|_{H_a^{l+2m}(Q)} + \|\xi R_{\mathcal{K}_1} f\|_{H_a^{l+2m}(Q)}) \leq c_3 \|f\|_{\mathcal{H}_a^l(Q, \Gamma)}. \quad (5.12)$$

Further, by virtue of (5.5), we have

$$\text{supp } \hat{\xi}^t(y', z') \subset B_{\delta(g^t)}.$$

Therefore,

$$\hat{\xi}^t \mathcal{L}'_{g^t} v = \hat{\xi}^t \mathcal{L}''_{g^t} v, \quad v \in \mathcal{H}_a^{l+2m}(\Theta). \quad (5.13)$$

It follows from Leibniz' formula that

$$\mathcal{L}'_{g^t}(\hat{\xi}^t v) = \hat{\xi}^t \mathcal{L}'_{g^t} v + \tilde{\mathcal{L}}_{g^t} v + \{0, \mathcal{T}_{j\rho\mu}^t v\}, \quad (5.14)$$

where $\tilde{\mathcal{L}}_{g^t} : \mathcal{H}_a^{l+2m-1}(\Theta) \rightarrow \mathcal{H}_a^l(\Theta, \Gamma)$ is a bounded operator such that $\text{supp } \tilde{\mathcal{L}}_{g^t} v \subset B_{\delta(g^t)}$, while

$$\mathcal{T}_{j\rho\mu}^t v = \sum_{(k,s) \neq (j,0)} (\hat{\xi}^t(\mathcal{G}_{j\rho ks} y', z') - \hat{\xi}^t(y', z'))(B_{j\rho\mu ks}^0(x', D)v_k)(\mathcal{G}_{j\rho ks} y', z')|_{\Gamma_{j\rho}}$$

and $\text{supp } \mathcal{T}_{j\rho\mu}^t v \subset B_{\delta(g^t)}$. Clearly,

$$\mathcal{T}_{j\rho\mu}^t : \mathcal{H}_a^{l+2m}(\Theta) \rightarrow H_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})$$

is a bounded operator. Moreover, since the function $\hat{\xi}^t$ does not depend on ε , it follows that

$$\|\tilde{\mathcal{L}}_{g^t} v\|_{\mathcal{H}_a^l(\Theta, \Gamma)} \leq c_4 \|v\|_{\mathcal{H}_a^{l+2m-1}(\Theta)}, \quad (5.15)$$

$$\|\mathcal{T}_{j\rho\mu}^t v\|_{H_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})} \leq c_5 \|v\|_{\mathcal{H}_a^{l+2m}(\Theta)}. \quad (5.16)$$

Using definition (5.9) of the operator $R_{\mathcal{K}_1}$, the isomorphisms U^t and F^t , and relations (5.14), (5.13), and (5.7), we obtain

$$\begin{aligned} \mathbf{L} R_{\mathcal{K}_1} f &= \sum_t (F^t)^{-1} \mathcal{L}'_{g^t} \hat{\xi}^t (\mathcal{L}''_{g^t})^{-1} F^t \left(\sum_q \xi_q^t f \right) \\ &= \sum_t (F^t)^{-1} \hat{\xi}^t F^t \left(\sum_q \xi_q^t f \right) + T_2 f + T_3 f \\ &= \sum_t (F^t)^{-1} F^t \left(\sum_q \hat{\xi}_q^t \xi_q^t f \right) + T_2 f + T_3 f \\ &= \sum_{t,q} \xi_q^t f + T_2 f + T_3 f, \end{aligned} \quad (5.17)$$

where

$$\begin{aligned} T_2 f &= \sum_t (F^t)^{-1} \tilde{\mathcal{L}}_{g^t} (\mathcal{L}''_{g^t})^{-1} F^t \left(\sum_q \xi_q^t f \right), \\ T_3 f &= \sum_t (F^t)^{-1} \left\{ 0, \mathcal{T}_{j\rho\mu}^t (\mathcal{L}''_{g^t})^{-1} F^t \left(\sum_q \xi_q^t f \right) \right\}. \end{aligned}$$

Since the operator $\tilde{\mathcal{L}}_{g^t} : \mathcal{H}_a^{l+2m-1}(\Theta) \rightarrow \mathcal{H}_a^l(\Theta, \Gamma)$ is bounded, $\text{supp } \tilde{\mathcal{L}}_{g^t} v \subset B_{\delta(g^t)}$, and $\mathcal{H}_a^{l+2m}(\Theta \cap B_{\delta(g^t)})$ is compactly embedded into $\mathcal{H}_a^{l+2m-1}(\Theta \cap B_{\delta(g^t)})$, it follows that the operator

$$T_2 : \mathcal{H}_a^l(Q, \Gamma) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$$

is compact. Furthermore, by (5.15), (5.16), and Corollary 2.1, we have

$$\|T_i f\|_{\mathcal{H}_a^l(Q, \Gamma)} \leq c_6 \|f\|_{\mathcal{H}_a^l(Q, \Gamma)}, \quad i = 2, 3 \quad (5.18)$$

(we have also used the fact that the operator \mathcal{L}''_{g^t} and the functions ξ_q^t do not depend on ε).

Now let us prove that the square of the operator T_3 is compact. Indeed,

$$\|(T_3)^2 f\|_{\mathcal{H}_a^l(Q, \Gamma)} \leq c_7 \sum_{t, j, \rho, \mu} \left\| \mathcal{T}_{j\rho\mu}^t (\mathcal{L}_{g^t}^{\prime\prime})^{-1} F^t \left(\sum_q \xi_q^t T_3 f \right) \right\|_{H_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})}. \quad (5.19)$$

It follows from (5.5) and (5.6) that

$$\text{supp} \left(\hat{\xi}^t (\mathcal{G}_{j\rho ks} y', z') - \hat{\xi}^t (y', z') \right) \subset B_{\delta(g^t)}$$

and

$$\hat{\xi}^t (\mathcal{G}_{j\rho ks} y', z') - \hat{\xi}^t (y', z') = 0$$

for $|y'| \leq \varepsilon'_1 / \chi_M$, where $\chi_M = \max_{j, \rho, k, s} \chi_{j\rho ks}$. Therefore, the condition $d_{k1} < d_{j\rho} + \varphi_{j\rho ks} < d_{k2}$ for $(k, s) \neq (j, 0)$ implies that

$$\|\mathcal{T}_{j\rho\mu}^t v\|_{H_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})} \leq c_8 \sum_{k=1}^{N(g^t)} \|v_k\|_{H_a^{l+2m}(\Omega_k^t)}, \quad (5.20)$$

where

$$\Omega_k^t = \{x' = (y', z') : d_{k1} + d_0 < \varphi < d_{k2} - d_0, |y'| > \varepsilon'_1 / \chi_M, |x'| < \delta(g^t)\}$$

and

$$d_0 = \min_{j, \rho, k, s} (d_{j\rho} + \varphi_{j\rho ks} - d_{k1}, d_{k2} - (d_{j\rho} + \varphi_{j\rho ks})) / 2 \quad ((k, s) \neq (j, 0)). \quad (5.21)$$

Using inequality (5.20), Lemma 3.5, and the equivalence of the norms in subspaces of the spaces $H_a^l(\Theta_k)$ and $W^l(\Theta_k)$ consisting of compactly supported functions vanishing near the edge \mathcal{P} , we obtain

$$\begin{aligned} & \left\| \mathcal{T}_{j\rho\mu}^t (\mathcal{L}_{g^t}^{\prime\prime})^{-1} F^t \left(\sum_q \xi_q^t T_3 f \right) \right\|_{H_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})} \\ & \leq c_9 \sum_k \left(\left\| A_k'' \left[(\mathcal{L}_{g^t}^{\prime\prime})^{-1} F^t \left(\sum_q \xi_q^t T_3 f \right) \right]_k \right\|_{H_a^l(\Theta_k)} \right. \\ & \quad \left. + \left\| \left[(\mathcal{L}_{g^t}^{\prime\prime})^{-1} F^t \left(\sum_q \xi_q^t T_3 f \right) \right]_k \right\|_{H_a^0(\Theta_k \cap B_{2\delta(g^t)})} \right), \end{aligned}$$

where

$$A_k'' = A_k(D_{x'}) + \eta(A_k^0(x', D_{x'}) - A_k(D_{x'})).$$

However, the first $N(g^t)$ components of the vector $F^t \left(\sum_q \xi_q^t T_3 f \right)$ are equal to zero. Therefore,

$$A_k'' \left[(\mathcal{L}_{g^t}^{\prime\prime})^{-1} F^t \left(\sum_q \xi_q^t T_3 f \right) \right]_k = 0, \quad k = 1, \dots, N(g^t), \quad (5.22)$$

and hence

$$\begin{aligned} & \left\| \mathcal{T}_{j\rho\mu}^t (\mathcal{L}_{g^t}^{\prime\prime})^{-1} F^t \left(\sum_q \xi_q^t T_3 f \right) \right\|_{H_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})} \\ & \leq c_9 \sum_k \left\| \left[(\mathcal{L}_{g^t}^{\prime\prime})^{-1} F^t \left(\sum_q \xi_q^t T_3 f \right) \right]_k \right\|_{H_a^0(\Theta_k \cap B_{2\delta(g^t)})}. \end{aligned} \quad (5.23)$$

Inequalities (5.19) and (5.23) and the compactness of the embedding $H_a^{l+2m}(\Theta_k) \subset H_a^0(\Theta_k \cap B_{2\delta(g^t)})$ imply that the operator T_3 has a compact square.

Similarly, one can show that the operator T_1 has a compact square. To this end, one must use the relation

$$\left[\overline{\omega_{is}(\Gamma_i)} \cap \left(\bigcup_{t, j} \hat{B}_{\delta(g_j^t)}(g_j^t) \right) \right] \setminus \mathcal{K}_1^{\varepsilon/2} \subset Q_b, \quad i = 1, \dots, N_0, \quad s = 1, \dots, S_i,$$

which holds for some $b > 0$.

Thus, it follows from (5.11) and (5.17) that

$$\mathbf{L}^1 R_{\mathcal{K}_1} f = \xi_0 f + T_{\mathcal{K}_1} f, \quad (5.24)$$

where

$$\xi_0(x) = \sum_{i, q} \xi_q^t(x) \quad (5.25)$$

and $T_{\mathcal{K}_1} : \mathcal{H}_a^l(Q, \Gamma) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ is a bounded operator with compact square. Moreover, inequalities (5.12) and (5.18) imply that

$$\|T_{\mathcal{K}_1} f\|_{\mathcal{H}_a^l(Q, \Gamma)} \leq c_{10} \|f\|_{\mathcal{H}_a^l(Q, \Gamma)}. \quad (5.26)$$

4. Take a number ε in the definition of the function ξ so small that

$$\mathcal{K}_1^{4\varepsilon} \subset \bigcup_{t,j} \hat{B}_{\delta(g^t)}(g_j^t)$$

and

$$\xi_0(x) \geq 1/2, \quad x \in \mathcal{K}_1^{4\varepsilon}$$

(the existence of such an ε follows from (5.8) and (5.25)). Later on, we will impose some additional conditions on ε .

For each ε , we consider a function $\zeta_0 \in C_0^\infty(\mathbb{R}^n)$ depending on ε , such that

$$\text{supp } \zeta_0 \subset \mathcal{K}_1^{4\varepsilon}; \quad \zeta_0(x) = 1, \quad x \in \mathcal{K}_1^{2\varepsilon}; \quad |D^\alpha \zeta_0(x)| \leq c_{11} \varepsilon^{-|\alpha|}. \quad (5.27)$$

For each point $g \in \overline{Q} \setminus \mathcal{K}_1^{2\varepsilon}$, we consider its $(\varepsilon/2)$ -neighborhood $B_{\varepsilon/2}(g)$. All these neighborhoods cover $\overline{Q} \setminus \mathcal{K}_1^{2\varepsilon}$. Choose a finite subcovering $\{B_{\varepsilon/2}(h^\tau)\}_{\tau=1}^{\tau_1}$, where $\tau_1 = \tau_1(\varepsilon)$. Let functions $\tilde{\zeta}^\tau \in C_0^\infty(\mathbb{R}^n)$ form a partition of unity for $\overline{Q} \setminus \mathcal{K}_1^{2\varepsilon}$, subordinated to the covering $\{B_{\varepsilon/2}(h^\tau)\}_{\tau=1}^{\tau_1}$. Then the functions

$$\zeta = \xi_0 + \zeta_0(1 - \xi_0), \quad \zeta^\tau = (1 - \zeta)\tilde{\zeta}^\tau, \quad \tau = 1, \dots, \tau_1, \quad (5.28)$$

form a partition of unity for \overline{Q} , subordinated to the covering by the sets $\bigcup_{t,j} \hat{B}_{\delta(g^t)}(g_j^t)$ and $B_{\varepsilon/2}(h^\tau)$, $\tau = 1, \dots, \tau_1$.

Due to Theorem 2.3 and to the general theory of elliptic boundary-value problems in the interior of a domain and near a smooth part of the boundary (see, e.g., [31]), there exist bounded operators

$$R_{\tau 0} : \{f \in \mathcal{H}_a^l(Q, \Gamma) : \text{supp } f \subset B_{\varepsilon/2}(h^\tau)\} \rightarrow \{u \in H_a^{l+2m}(Q) : \text{supp } u \subset B_\varepsilon(h^\tau)\}$$

and compact operators

$$T_{\tau 0} : \{f \in \mathcal{H}_a^l(Q, \Gamma) : \text{supp } f \subset B_{\varepsilon/2}(h^\tau)\} \rightarrow \{f \in \mathcal{H}_a^l(Q, \Gamma) : \text{supp } f \subset B_\varepsilon(h^\tau)\}$$

such that

$$\mathbf{L}^0 R_{\tau 0} f = f + T_{\tau 0} f. \quad (5.29)$$

For any $f \in \mathcal{H}_a^l(Q, \Gamma)$, we set

$$Rf = R_{\mathcal{K}_1} f + R_{\mathcal{K}_1}(\eta f) + \sum_{\tau} R_{\tau 0}(\zeta^\tau f), \quad (5.30)$$

where $\eta(x) = \zeta_0(x)(1 - \xi_0(x))/\xi_0(x)$ for $x \in \mathcal{K}_1^{4\varepsilon}$ and $\eta(x) = 0$ for $x \notin \mathcal{K}_1^{4\varepsilon}$. Note that $\text{supp } \zeta_0 \subset \mathcal{K}_1^{4\varepsilon}$ and $\xi_0(x) \geq 1/2$ for $x \in \mathcal{K}_1^{4\varepsilon}$; hence, the function η is supported on $\mathcal{K}_1^{4\varepsilon}$ and infinitely differentiable on \mathbb{R}^n . We have

$$\mathbf{L}^1 Rf = \mathbf{L}^1 R_{\mathcal{K}_1} f + \mathbf{L}^1 R_{\mathcal{K}_1}(\eta f) + \sum_{\tau} \mathbf{L}^0 R_{\tau 0}(\zeta^\tau f) + \sum_{\tau} \{0, B_{i\mu}^1 R_{\tau 0}(\zeta^\tau f)\}. \quad (5.31)$$

Since $\zeta(x) = 1$ for $x \in \mathcal{K}_1^{2\varepsilon}$, it follows that $\text{supp } \zeta^\tau f \subset \overline{Q} \setminus \mathcal{K}_1^{2\varepsilon}$. Thus, we see that $\text{supp } R_{\tau 0}(\zeta^\tau f) \subset \overline{Q} \setminus \mathcal{K}_1^\varepsilon$, while $\text{supp } \xi \subset \mathcal{K}_1^\varepsilon$. Therefore,

$$B_{i\mu}^1 R_{\tau 0}(\zeta^\tau f) = 0.$$

Combining this relation with equalities (5.24), (5.29), and (5.31), we obtain

$$\mathbf{L}^1 Rf = \xi_0 f + T_{\mathcal{K}_1} f + \zeta_0(1 - \xi_0)f + T_{\mathcal{K}_1}(\eta f) + \sum_{\tau} \zeta^\tau f + Tf = f + T_{\mathcal{K}_1} f + Mf + Tf, \quad (5.32)$$

where

$$Mf = T_{\mathcal{K}_1}(\eta f),$$

while $T : \mathcal{H}_a^l(Q, \Gamma) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ is a compact operator (whose norm may increase as $\varepsilon \rightarrow 0$). By (5.26), we have

$$\|Mf\|_{\mathcal{H}_a^l(Q, \Gamma)} \leq c_{10} \|\eta f\|_{\mathcal{H}_a^l(Q, \Gamma)}.$$

However, $(1 - \xi_0(x))/\xi_0(x) = 0$ for $x \in \mathcal{K}_1$ and the function ζ_0 is supported in $\mathcal{K}_1^{4\varepsilon}$ and satisfies the inequality in (5.27). Therefore, it follows from the last estimate, from Lemmas 4.1 and 4.2 and from Remark 4.1 that

$$\|Mf\|_{\mathcal{H}_a^l(Q, \Gamma)} \leq c_{11}\varepsilon\|f\|_{\mathcal{H}_a^l(Q, \Gamma)}. \quad (5.33)$$

It follows from (5.26), (5.33), and from Lemma 5.1 that the operator $\mathbf{I} + T_{\mathcal{K}_1} + M$ has the Fredholm property, provided that $\varepsilon > 0$ is sufficiently small. Applying Theorem 16.4 in [18], we see that the operator

$$\mathbf{L}^1 R = \mathbf{I} + T_{\mathcal{K}_1} + M + T$$

also has the Fredholm property. Therefore, by Theorem 15.2 in [18], there exist a bounded operator $R' : \mathcal{H}_a^l(Q, \Gamma) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ and a compact operator $\mathbf{T}_1 : \mathcal{H}_a^l(Q, \Gamma) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ such that

$$\mathbf{L}^1 R R' = \mathbf{I} + \mathbf{T}_1.$$

Setting $\mathbf{R}_1 = R R'$, we complete the proof. \square

$$\text{Set } \mathcal{H}_a^l(\partial Q) = \prod_{i, \mu} H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i).$$

To construct a right regularizer for the operator \mathbf{L} , we also need to prove the existence of a “right regularizer” \mathbf{R}'_1 for the operator \mathbf{L}^1 , which is defined on the functions $f' \in \mathcal{H}_a^l(\partial Q)$ and possesses the following properties: $\mathbf{R}'_1 f'$ is supported near the boundary ∂Q for all f' and $\mathbf{R}'_1 f'$ is supported near the set \mathcal{K}_1 for f' supported near \mathcal{K}_1 .

Lemma 5.3. *Let the conditions of Theorem 5.1 be fulfilled. Then there exist a linear bounded operator $\mathbf{R}'_1 : \mathcal{H}_a^l(\partial Q) \rightarrow H_a^{l+2m}(Q)$ and a compact operator $\mathbf{T}'_1 : \mathcal{H}_a^l(\partial Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ such that*

$$\mathbf{L}^1 \mathbf{R}'_1 f' = \{0, f'\} + \mathbf{T}'_1 f', \quad (5.34)$$

$$\text{supp } \mathbf{R}'_1 f' \subset \overline{Q} \setminus Q_\sigma \quad (5.35)$$

for any $f' \in \mathcal{H}_a^l(\partial Q)$, and

$$\text{supp } \mathbf{R}'_1 f' \subset \mathcal{K}_1^{2\kappa} \quad (5.36)$$

for $f' \in \mathcal{H}_a^l(\partial Q)$, $\text{supp } f' \subset \overline{\mathcal{K}_1^\kappa}$, where $\kappa, \sigma > 0$ are the constants from Lemma 4.6.

Proof. 1. Fix an arbitrary number $\hat{\varepsilon} > 0$ independent of ε . Similarly to the proof of Lemma 5.2, we can construct functions $\hat{\zeta}, \hat{\zeta}^\tau \in C_0^\infty(\mathbb{R}^n)$, $\tau = 1, \dots, \hat{\tau}_1$, $\hat{\tau}_1 = \hat{\tau}_1(\hat{\varepsilon})$, which form a partition of unity for \overline{Q} , subordinated to the covering by the sets $\mathcal{K}_1^{2\hat{\varepsilon}}$ and $B_{\hat{\varepsilon}/2}(h^\tau)$, $\tau = 1, \dots, \hat{\tau}_1$, where $h^\tau \in \overline{Q} \setminus \mathcal{K}_1^{2\hat{\varepsilon}}$. In particular, the function $\hat{\zeta}$ can be chosen in such a way that

$$\hat{\zeta}(x) = 1, \quad x \in \mathcal{K}_1^{\hat{\varepsilon}}. \quad (5.37)$$

2. Due to Theorem 2.3 and to the general theory of elliptic boundary-value problems near a smooth part of the boundary (see, e.g., [31]), there exist bounded operators

$$R'_{\tau 0} : \{f' \in \mathcal{H}_a^l(\partial Q) : \text{supp } f' \subset B_{\hat{\varepsilon}/2}(h^\tau)\} \rightarrow \{u \in H_a^{l+2m}(Q) : \text{supp } u \subset B_{\hat{\varepsilon}}(h^\tau)\}$$

and compact operators

$$T'_{\tau 0} : \{f' \in \mathcal{H}_a^l(\partial Q) : \text{supp } f' \subset B_{\hat{\varepsilon}/2}(h^\tau)\} \rightarrow \{f \in \mathcal{H}_a^l(Q, \Gamma) : \text{supp } f \subset B_{\hat{\varepsilon}}(h^\tau)\}$$

such that

$$\mathbf{L}^0 R'_{\tau 0} f' = \{0, f'\} + T'_{\tau 0} f'. \quad (5.38)$$

For any $f' \in \mathcal{H}_a^l(\partial Q)$, we set

$$\mathbf{R}'_1 f' = \hat{\zeta} u + \sum_{\tau} u^\tau, \quad (5.39)$$

where

$$u = \mathbf{R}_1 \{0, f'\}, \quad u^\tau = R'_{\tau 0}(\hat{\zeta}^\tau \{0, f'\}).$$

Clearly, property (5.35) holds for $2\hat{\varepsilon} < \sigma$, while property (5.36) holds for $2\hat{\varepsilon} < 2\kappa$ and $\kappa + \hat{\varepsilon}/2 < 2\kappa$. Let us prove relation (5.34).

Using (5.39) and Leibniz' formula, we have

$$\mathbf{L}^1 \mathbf{R}'_1 f' = \hat{\zeta} \mathbf{L}^1 u + \left\{0, \sum_{s=1}^{S_i} T_{i\mu s} f'\right\} + T u + \sum_{\tau} \mathbf{L}^0 u^\tau + \{0, B_{i\mu}^1 u^\tau\}, \quad (5.40)$$

where

$$T_{i\mu s} f' = (\hat{\zeta}(\omega_{is}(x)) - \hat{\zeta}(x))(B_{i\mu s}^0(x, D_x)(\xi u))(\omega_{is}(x))|_{\Gamma_i}, \quad (5.41)$$

while $T : H_a^{l+2m-1}(Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ is a bounded operator. By virtue of the compactness of the embedding $H_a^{l+2m}(Q) \subset H_a^{l+2m-1}(Q)$, the operator $T : H_a^{l+2m}(Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ is compact.

Now it follows from Lemma 5.2, from (5.38), and from (5.40) that

$$\mathbf{L}^1 \mathbf{R}'_1 f' = \{0, f'\} + T' f' + \left\{0, \sum_{s=1}^{S_i} T_{i\mu s} f'\right\} + \{0, B_{i\mu}^1 u^\tau\}, \quad (5.42)$$

where $T' : \mathcal{H}_a^l(\partial Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ is a compact operator.

3. Let us prove that the operator $T_{i\mu s} : \mathcal{H}_a^l(\partial Q) \rightarrow H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)$ is compact. Since ω_{is} are C^∞ diffeomorphisms, it follows from Lemma 4.5 that

$$\begin{aligned} \|T_{i\mu s} f'\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \\ \leq k_1 \left\| \left(\hat{\zeta}(x) - \hat{\zeta}(\omega_{is}^{-1}(x)) \right) (B_{i\mu s}^0(x, D_x)(\xi u)) \right\|_{\omega_{is}(\Gamma_i)} \Big\|_{H_a^{l+2m-m_{i\mu}-1/2}(\omega_{is}(\Gamma_i))}, \end{aligned} \quad (5.43)$$

where $k_1, k_2, \dots > 0$ do not depend on f' .

Denote $\mathcal{M}_{is} = \omega_{is}(\Gamma_i) \setminus \omega_{is}(\Gamma_i)$. For every $x \in \mathcal{M}_{is}$, we have either $x \in \mathcal{K}_1$, or $x \in \mathcal{K}_2$, or $x \in \mathcal{K}_3$. If $x \in \mathcal{M}_{is} \cap \mathcal{K}_1$, then both x and $\omega_{is}^{-1}(x)$ belong to \mathcal{K}_1 . Therefore,

$$\hat{\zeta}(x) - \hat{\zeta}(\omega_{is}^{-1}(x)) = 0, \quad x \in (\mathcal{M}_{is} \cap \mathcal{K}_1)^d,$$

where $(\mathcal{M}_{is} \cap \mathcal{K}_1)^d$ is the d -neighborhood of the set $\mathcal{M}_{is} \cap \mathcal{K}_1$ and $d > 0$ is sufficiently small.

If $\mathcal{M}_{is} \cap (\mathcal{K}_2 \cup \mathcal{K}_3) \neq \emptyset$, then

$$\xi(x) = 0, \quad x \in (\mathcal{M}_{is} \cap (\mathcal{K}_2 \cup \mathcal{K}_3))^d,$$

where $(\mathcal{M}_{is} \cap (\mathcal{K}_2 \cup \mathcal{K}_3))^d$ is the d -neighborhood of the set $\mathcal{M}_{is} \cap (\mathcal{K}_2 \cup \mathcal{K}_3)$ and $d > 0$ is sufficiently small.

In all these cases, we see that

$$\text{supp} \left(\hat{\zeta}(x) - \hat{\zeta}(\omega_{is}^{-1}(x)) \right) (B_{i\mu s}^0(x, D_x)(\xi u)) \Big|_{\omega_{is}(\Gamma_i)} \subset Q_b,$$

where $b > 0$ is sufficiently small.

Using estimate (5.43), Lemma 3.5, and equivalence of the norms in $H_a^l(Q_b)$ and $W^l(Q_b)$, we obtain

$$\|T_{i\mu s} f'\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq k_2 \|u\|_{W^{l+2m}(Q_b)} \leq k_3 (\|A^0 u\|_{H_a^l(Q)} + \|u\|_{H_a^0(Q)}). \quad (5.44)$$

By Lemma 5.2, $A^0 u = A^0 \mathbf{R}^1 \{0, f'\} = \mathbf{T}_{11} f'$, where $\mathbf{T}_{11} : \mathcal{H}_a^l(\partial Q) \rightarrow H_a^l(Q)$ is a compact operator. Hence, inequality (5.44) takes the form

$$\|T_{i\mu s} f'\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq k_3 (\|\mathbf{T}_{11} f'\|_{H_a^l(Q)} + \|\mathbf{R}^1 \{0, f'\}\|_{H_a^0(Q)}),$$

which implies that $T_{i\mu s}$ is a compact operator (because \mathbf{T}_{11} is compact and $H_a^{l+2m}(Q)$ is compactly embedded into $H_a^0(Q)$).

4. The expression $B_{i\mu}^1 u^\tau$ consists of the terms

$$(B_{i\mu s}^0(x, D_x)(\xi R'_{\tau 0}(\hat{\zeta}^\tau \{0, f'\}))) (\omega_{is}(x)) \Big|_{\Gamma_i}, \quad s = 1, \dots, S_i. \quad (5.45)$$

Since $\text{supp } R'_{\tau 0}(\hat{\zeta}^\tau \{0, f'\}) \subset B_\varepsilon(h^\tau)$, where $h^\tau \in \overline{Q} \setminus \mathcal{K}_1^{2\varepsilon}$, it follows that

$$\text{supp } R'_{\tau 0}(\hat{\zeta}^\tau \{0, f'\}) \subset \overline{Q} \setminus \mathcal{K}_1^\varepsilon.$$

On the other hand, $\text{supp } \xi \subset \mathcal{K}_1^\varepsilon$. Hence,

$$\text{supp } B_{i\mu s}^0(x, D_x)(\xi R'_{\tau 0}(\hat{\zeta}^\tau \{0, f'\})) \Big|_{\omega_{is}(\Gamma_i)} \subset Q_b,$$

where $b > 0$ is sufficiently small. Therefore, applying Lemma 3.5 and equality (5.38), we can show similarly to the above that each of the operators in (5.45) is compact. Thus, we see that (5.42) is equivalent to (5.34). \square

Now we can prove Theorem 5.2.

Proof of Theorem 5.2. 1. We set

$$\Phi = B^2 \mathbf{R}_1 f, \quad f = \{f_0, f'\} \in \mathcal{H}_a^l(Q, \Gamma). \quad (5.46)$$

Introduce the bounded operator $\mathbf{R} : \mathcal{H}_a^l(Q, \Gamma) \rightarrow H_a^{l+2m}(Q)$ by the formula

$$\mathbf{R}f = \mathbf{R}_1 f - \mathbf{R}'_1 \Phi + \mathbf{R}'_1 B^2 \mathbf{R}'_1 \Phi,$$

where \mathbf{R}_1 and \mathbf{R}'_1 are the operators occurring in Lemmas 5.2 and 5.3, respectively. Let us show that \mathbf{R} is the desired operator.

For simplicity, we denote diverse compact operators by the same letter T .

It follows from Lemmas 5.2 and 5.3 that

$$A\mathbf{R}f = A^0 \mathbf{R}f = A^0 \mathbf{R}_1 f - A^0 \mathbf{R}'_1 (\Phi - B^2 \mathbf{R}'_1 \Phi) = f_0 + Tf \quad (5.47)$$

(recall that $A^1 = 0$) and

$$\begin{aligned} C\mathbf{R}f &= C\mathbf{R}_1 f - C\mathbf{R}'_1 \Phi + C\mathbf{R}'_1 B^2 \mathbf{R}'_1 \Phi \\ &= (f' + Tf) - (\Phi + T\Phi) + (B^2 \mathbf{R}'_1 \Phi + TB^2 \mathbf{R}'_1 \Phi) = f' - \Phi + B^2 \mathbf{R}'_1 \Phi + Tf. \end{aligned} \quad (5.48)$$

Applying the operator B^2 to the function $\mathbf{R}f$ and using (5.46), we obtain

$$B^2 \mathbf{R}f = \Phi - B^2 \mathbf{R}'_1 \Phi + B^2 \mathbf{R}'_1 B^2 \mathbf{R}'_1 \Phi. \quad (5.49)$$

Summing relations (5.48) and (5.49) and recalling that $B^3 = 0$, we obtain

$$B\mathbf{R}f = f' + Tf + B^2 \mathbf{R}'_1 B^2 \mathbf{R}'_1 \Phi. \quad (5.50)$$

2. Let us show that

$$B^2 \mathbf{R}'_1 B^2 \mathbf{R}'_1 \Phi = 0. \quad (5.51)$$

It follows from relation (5.35) in Lemma 5.3 that

$$\text{supp } \mathbf{R}'_1 \Phi \subset \overline{Q} \setminus Q_\sigma.$$

Therefore, estimate (4.25) implies that

$$\text{supp } B^2 \mathbf{R}'_1 \Phi \subset \overline{\mathcal{K}_1^\kappa}.$$

Furthermore, it follows from relation (5.36) in Lemma 5.3 that

$$\text{supp } \mathbf{R}'_1 B^2 \mathbf{R}'_1 \Phi \subset \mathcal{K}_1^{2\kappa}.$$

Combining this fact with inequality (4.24) yields (5.51).

Relations (5.47), (5.50), and (5.51) prove the theorem. \square

Remark 5.1. Using the results in [7], one can show that Theorem 5.1 remains true for the case in which the transformations ω_{is} are nonlinear near the set \mathcal{K}_1 , while their linear parts at the points of the set \mathcal{K}_1 satisfy Condition 1.4. Moreover, the index of the problem with nonlinear transformations ω_{is} is equal to the index of the corresponding problem with transformations linearized near the set \mathcal{K}_1 .

6 Some Generalizations

6.1

In this section, we generalize the results of Secs. 4 and 5 to the case where diffeomorphisms ω_{is} are defined only on some neighborhood of the set \mathcal{K}_1 , the operators $B_{i\mu}^2$ are abstract nonlocal operators supported outside the set \mathcal{K}_1 , and A^1 and $B_{i\mu}^3$ are compact perturbations on the corresponding weighted spaces.

Consider the differential operators

$$A^0 \equiv A^0(x, D) = \sum_{|\alpha|=2m} a_\alpha(x) D^\alpha, \quad B_{i\mu s}^0 \equiv B_{i\mu s}^0(x, D) = \sum_{|\alpha|=m_{i\mu}} b_{i\mu s \alpha}(x) D^\alpha,$$

where $a_\alpha, b_{i\mu s \alpha} \in C^\infty(\mathbb{R}^n)$ are complex-valued functions ($i = 1, \dots, N_0$; $\mu = 1, \dots, m$; $s = 0, \dots, S_i$), $m_{i\mu} \leq 2m - 1$.

Let a domain $Q \subset \mathbb{R}^n$ satisfy the assumptions of Sec. 1. As in Sec. 1, we denote

$$\mathcal{K}_1 = \bigcup_{\nu=1}^{N_1} \mathcal{K}_{1\nu} = \partial Q \setminus \bigcup_i \Gamma_i,$$

where $\mathcal{K}_{1\nu}$ are mutually disjoint $(n-2)$ -dimensional connected C^∞ manifolds without a boundary.

Let ω_{is} ($i = 1, \dots, N_0$; $s = 1, \dots, S_i$) denote a C^∞ diffeomorphism mapping $(\overline{\Gamma_i} \setminus \Gamma_i)^{\varepsilon_0}$ onto the set $\omega_{is}((\overline{\Gamma_i} \setminus \Gamma_i)^{\varepsilon_0})$, where $\varepsilon_0 > 0$ is some number, in such a way that

1. $\omega_{is}(\Gamma_i \cap \mathcal{K}_1^{\varepsilon_0}) \subset Q$,
2. if η ($1 \leq \eta \leq N_1$) and i ($1 \leq i \leq N_0$) are such that $\mathcal{K}_{1\eta} \subset \overline{\Gamma_i} \setminus \Gamma_i$, then, for every s ($1 \leq s \leq S_i$), there is ν ($1 \leq \nu \leq N_1$) such that $\omega_{is}(\mathcal{K}_{1\eta}) = \mathcal{K}_{1\nu}$.

Along with the set \mathcal{K}_1 , we introduce the set

$$K_2 = \bigcup_{\nu=1}^{N_2} K_{2\nu} \subset \bigcup_i \Gamma_i,$$

where $K_{2\nu}$ are mutually disjoint connected C^∞ manifolds without a boundary. In particular, the set K_2 can be empty. We use either the set

$$K = \mathcal{K}_1$$

or the set

$$K = \mathcal{K}_1 \cup K_2$$

in the definition of the space $H_a^l(Q) = H_a^l(Q, K)$.

We study the following nonlocal elliptic problem:

$$Au \equiv A^0 u + A^1 u = f_0(x), \quad x \in Q, \quad (6.1)$$

$$B_{i\mu} u \equiv \sum_{j=0}^3 B_{i\mu}^j u = f_{i\mu}(x), \quad x \in \Gamma_i; \quad i = 1, \dots, N_0; \quad \mu = 1, \dots, m. \quad (6.2)$$

Here

$$B_{i\mu}^0 = B_{i\mu 0}^0 u|_{\Gamma_i}, \quad B_{i\mu}^1 u = \sum_{s=1}^{S_i} (B_{i\mu s}^0(x, D)(\xi u))(\omega_{is}(x))|_{\Gamma_i},$$

a function $\xi \in C_0^\infty(\mathbb{R}^n)$ is such that $\xi(x) = 1$ for $x \in \mathcal{K}_1^{\varepsilon/2}$ and $\text{supp } \xi \subset K_1^\varepsilon$, while $\varepsilon > 0$ is so small that if $\omega_{is}(\mathcal{K}_{1\eta}) = \mathcal{K}_{1\nu}$, then $\mathcal{K}_{1\nu}^\varepsilon \subset \omega_{is}(\mathcal{K}_{1\eta}^{\varepsilon_0})$.

Assume that the operators A^0 and $B_{i\mu 0}^0$ satisfy Conditions 1.1 and 1.2, and the transformations ω_{is} satisfy Conditions 1.3 and 1.4. We also suppose that the following conditions for the operators A^1 , $B_{i\mu}^2$, and $B_{i\mu}^3$ hold.

Condition 6.1 (compactness of perturbations). *The linear operators*

$$A^1 : H_a^{l+2m-1}(Q) \rightarrow H_a^l(Q), \quad B_{i\mu}^3 : H_a^{l+2m-1}(Q) \rightarrow H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)$$

are bounded ($i = 1, \dots, N_0$; $\mu = 1, \dots, m$).

Condition 6.2 (separability from the conjugation points). *The linear operators $B_{i\mu}^2 : H_a^{l+2m}(Q) \rightarrow H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)$ are bounded, and there exist numbers $\sigma > 0$ and $\varkappa_1 > \varkappa_2 > 0$ such that*

$$\|B_{i\mu}^2 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq c_1 \|u\|_{H_a^{l+2m}(Q \setminus \overline{\mathcal{K}_1^{\varkappa_1}})} \quad (6.3)$$

for all $u \in H_a^{l+2m}(Q \setminus \overline{\mathcal{K}_1^{\varkappa_1}})$ and

$$\|B_{i\mu}^2 u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i \setminus \overline{\mathcal{K}_1^{\varkappa_2}})} \leq c_2 \|u\|_{H_a^{l+2m}(Q_\sigma)} \quad (6.4)$$

for all $u \in H_a^{l+2m}(Q_\sigma)$; here $i = 1, \dots, N_0$; $\mu = 1, \dots, m$; $c_1, c_2 > 0$ do not depend on u .

Remark 6.1. It follows from Lemma 4.6 that problem (1.3), (1.4) can be represented in the form (6.1), (6.2) with the operators A^1 , $B_{i\mu}^2$, and $B_{i\mu}^3$ satisfying Conditions 6.1 and 6.2.

Remark 6.2. The proofs of Theorems 4.1 and 5.2 use only Conditions 6.1 and 6.2, rather than any explicit form of the operators A^1 , $B_{i\mu}^2$, and $B_{i\mu}^3$. To prove that those operators satisfied Conditions 6.1 and 6.2, we used Condition 4.1, which provided the choice of the set K and the number a in Secs. 1–5. However, Condition 4.1 is needless in this section because the fulfilment of Conditions 6.1 and 6.2 is postulated rather than proved.

6.2

We introduce the linear bounded operator

$$\mathbf{L} = \{A, B_{i\mu}\} : H_a^{l+2m}(Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$$

corresponding to problem (6.1), (6.2).

For every fixed point $g \in \mathcal{K}_1$, we consider the linear bounded operator $\mathcal{L}_g(\omega) : \mathcal{E}_a^{l+2m}(\theta) \rightarrow \mathcal{E}_a^l(\theta, \gamma)$ given by (2.1), where $\omega \in S^{n-3}$. We also consider the analytic operator-valued function $\hat{\mathcal{L}}_g(\lambda) : \mathcal{W}^{l+2m}(d_1, d_2) \rightarrow \mathcal{W}^l[d_1, d_2]$ given by (2.2).

For every fixed point $g \in K_2$, we consider the linear bounded operator $\mathcal{L}_g(\omega) : E_a^{l+2m}(\mathbb{R}_+^2) \rightarrow \mathcal{E}_a^l(\mathbb{R}_+^2, \gamma)$ given by (2.26), where $\omega \in S^{n-3}$. We also consider the analytic operator-valued function $\hat{\mathcal{L}}_g(\lambda) : W^{l+2m}(-\pi/2, \pi/2) \rightarrow \mathcal{W}^l[-\pi/2, \pi/2]$ given by (2.27).

Similarly to the proofs of Theorems 4.1 and 5.2, using Conditions 6.1 and 6.2 and assuming that the number $\varepsilon > 0$ in the definition of the function ξ is sufficiently small, we obtain the following two results.

Theorem 6.1. *Let Conditions 1.1–1.4, 6.1, and 6.2 hold. Assume that the line $\text{Im } \lambda = a + 1 - l - 2m$ contains no eigenvalues of $\hat{\mathcal{L}}_g(\lambda)$ for any $g \in K$ and $\dim \mathcal{N}(\mathcal{L}_g(\omega)) = \text{codim } \mathcal{R}(\mathcal{L}_g(\omega)) = 0$ for any $g \in K$ and $\omega \in S^{n-3}$. Then the following estimate holds for all $u \in H_a^{l+2m}(Q)$:*

$$\|u\|_{H_a^{l+2m}(Q)} \leq c(\|\mathbf{L}u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \|u\|_{H_a^{l+2m-1}(Q)}),$$

where $c > 0$ does not depend on u .

Theorem 6.2. *Let the conditions of Theorem 6.1 be fulfilled. Then there exists a linear bounded operator $\mathbf{R} : \mathcal{H}_a^l(Q, \Gamma) \rightarrow H_a^{l+2m}(Q)$ such that*

$$\mathbf{L}\mathbf{R} = \mathbf{I} + \mathbf{T}$$

where $\mathbf{T} : \mathcal{H}_a^l(Q, \Gamma) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ is a compact operator.

Along with the operator \mathbf{L} , we consider the bounded operator

$$\mathbf{L}^1 = \{A^0, B_{i\mu}^0 + B_{i\mu}^1\} : H_a^{l+2m}(Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma).$$

It follows from Theorem 5.1 that the operator \mathbf{L}^1 has the Fredholm property.

Similarly to the proofs of Theorem 5.1 and Corollary 5.1, using Theorems 6.1 and 6.2, we obtain the following result.

Theorem 6.3. *Let the conditions of Theorem 6.1 be fulfilled. Then the operator $\mathbf{L} : H_a^{l+2m}(Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ has the Fredholm property and $\text{ind } \mathbf{L} = \text{ind } \mathbf{L}^1$.*

Theorem 6.3 shows that the addition of the operators A^1 , $B_{i\mu}^2$, and $B_{i\mu}^3$ satisfying Conditions 6.1 and 6.2 neither violates the Fredholm property nor changes the index.

6.3

Now we consider an example of an elliptic problem with distributed nonlocal terms satisfying Condition 6.2.

Example 6.1. Let $Q \subset \mathbb{R}^3$ be a bounded domain with boundary ∂Q which is a surface of revolution about the axis x_3 . Denote $P = \{0, 0, 1\} \cup \{0, 0, -1\}$. Assume that, outside $P^{1/4}$, the surface ∂Q coincides with the boundary of the domain

$$\{x : x_3 < 1 - \sqrt{x_1^2 + x_2^2}\} \cap \{x : x_3 > -1 + \sqrt{x_1^2 + x_2^2}\}.$$

Denote

$$\Gamma_1 = \{x \in \partial Q : x_3 < 0\}, \quad \Gamma_2 = \{x \in \partial Q : x_3 > 0\}.$$

In this case, we have

$$\mathcal{K}_1 = \{x : x_1^2 + x_2^2 = 1, x_3 = 0\}.$$

Assume that the boundary ∂Q is infinitely smooth outside the set \mathcal{K}_1 .

We introduce the operators

$$B_l^1 u = -\alpha_l(\xi u)(\omega_l(x))|_{\Gamma_l}, \quad l = 1, 2. \tag{6.5}$$

The transformations $\omega_l(x)$ in (6.5) are defined for $x \in \mathcal{K}_1^{\varepsilon_0}$ by the formula

$$\begin{aligned} \omega_l(x) = & \left(\cos \varphi \left[1 - \frac{1}{\sqrt{2}}((1-r) + (-1)^l x_3) \right], \right. \\ & \left. \sin \varphi \left[1 - \frac{1}{\sqrt{2}}((1-r) + (-1)^l x_3) \right], \frac{1}{\sqrt{2}}[(-1)^{l+1}(1-r) + x_3] \right), \end{aligned} \quad (6.6)$$

where r, φ, x_3 are the cylindrical coordinates of the point x , the number $\varepsilon_0 > 0$ is sufficiently small, the function $\xi \in C_0^\infty(\mathbb{R}^3)$ is such that $\xi(x) = 1$ for $x \in \mathcal{K}_1^{\varepsilon/2}$ and $\text{supp } \xi \subset \mathcal{K}_1^\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$, and $\alpha_1, \alpha_2 \in \mathbb{R}$. Clearly, we have $\omega_l(\mathcal{K}_1) = \mathcal{K}_1$.

Since $\Gamma_l \in C^\infty$, one can find a sufficiently small $\varkappa > 0$ possessing the following property: for any $x \in \Gamma_l^{5\varkappa} \cap Q$, there exists a unique pair (y, t) , $y \in \Gamma_l$, $t > 0$, such that $x = y + n_y t$, where n_y denotes the unit normal to Γ_l at the point y , directed inside the domain Q . One can show that the transformation $x \mapsto (y, t)$ is a C^∞ diffeomorphism mapping $\Gamma_l^{5\varkappa} \cap Q$ onto $\Gamma_l \times (0, 5\varkappa)$, provided that $\varkappa > 0$ is sufficiently small.

Introduce a function $\eta \in C_0^\infty(\mathbb{R})$ such that $\eta(t) = 1$ for $t \in (3\varkappa, 4\varkappa)$ and $\text{supp } \eta \subset (2\varkappa, 5\varkappa)$. Consider the operators

$$B_l^2 u = F^{-1}(b_l F(\eta u))|_{t=0}, \quad l = 1, 2, \quad (6.7)$$

where

$$F(\eta u)(y, \lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\lambda t} \eta(t) u(y, t) dt$$

is the Fourier transform with respect to t , F^{-1} is the inverse Fourier transform, $b_l(\lambda)$ is a function continuous on \mathbb{R} and such that

$$\sup_{\lambda \in \mathbb{R}} |b_l(\lambda)| < \infty. \quad (6.8)$$

We consider the following nonlocal boundary-value problem:

$$-\Delta u = f_0(x), \quad x \in Q, \quad (6.9)$$

$$u|_{\Gamma_l} + B_l^1 u + B_l^2 u = f_l(x), \quad x \in \Gamma_l, \quad l = 1, 2. \quad (6.10)$$

We set $K = \mathcal{K}_1$ in the definition of the space $H_a^l(Q) = H_a^l(Q, K)$.

The Jacobian $\frac{D\omega_l}{Dx}$ can be calculated on \mathcal{K}_1 as follows:

$$\left. \frac{D\omega_l}{Dx} \right|_{\mathcal{K}_1} = \left. \frac{D\omega_l}{D(r, \varphi, x_3)} \right|_{\mathcal{K}_1} \left. \frac{D(r, \varphi, x_3)}{Dx} \right|_{\mathcal{K}_1}.$$

Since $\left. \frac{D(r, \varphi, x_3)}{Dx} \right|_{\mathcal{K}_1} = 1$, we have

$$\begin{aligned} \left. \frac{D\omega_l}{Dx} \right|_{\mathcal{K}_1} &= \det \begin{pmatrix} \frac{\partial \omega_{l1}}{\partial r} & \frac{\partial \omega_{l1}}{\partial \varphi} & \frac{\partial \omega_{l1}}{\partial x_3} \\ \frac{\partial \omega_{l2}}{\partial r} & \frac{\partial \omega_{l2}}{\partial \varphi} & \frac{\partial \omega_{l2}}{\partial x_3} \\ \frac{\partial \omega_{l3}}{\partial r} & \frac{\partial \omega_{l3}}{\partial \varphi} & \frac{\partial \omega_{l3}}{\partial x_3} \end{pmatrix} \Big|_{\mathcal{K}_1} \\ &= \det \begin{pmatrix} \frac{1}{\sqrt{2}} \cos \varphi & -\sin \varphi & \frac{1}{\sqrt{2}}(-1)^{l+1} \cos \varphi \\ \frac{1}{\sqrt{2}} \sin \varphi & \cos \varphi & \frac{1}{\sqrt{2}}(-1)^{l+1} \sin \varphi \\ \frac{1}{\sqrt{2}}(-1)^l & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = 1. \end{aligned}$$

Therefore, we can choose $\varepsilon_0 > 0$ so small that $\omega_l : \mathcal{K}_1^{\varepsilon_0} \rightarrow \omega_l(\mathcal{K}_1^{\varepsilon_0})$ is a C^∞ diffeomorphism. Furthermore, $\omega_l(\mathcal{K}_1^{\varepsilon_0}) = \mathcal{K}_1^{\varepsilon_0}$ and $\omega_l(\Gamma_l \cap \mathcal{K}_1^{\varepsilon_0}) = \{x : 1 - \varepsilon_0 < r < 1, x_3 = 0\} \subset Q$.

Since $\omega_l(g) = g$ for $g \in \mathcal{K}_1$ and $l = 1, 2$, the orbit of each point $g \in \mathcal{K}_1$ consists of one point $g_1 = g$. For $x \neq 0$, we introduce the new variables $x' = (y', z')$ given by

$$y'_1 = 1 - r, \quad y'_2 = x_3, \quad z' = \varphi - \varphi_1,$$

where $y' = (y'_1, y'_2)$, while r, φ, x_3 and $1, \varphi_1, 0$ are the cylindrical coordinates of the points x and g_1 , respectively.

Set $V(0) = \{x' : |y'| < \varepsilon_1, |z'| < \varepsilon_1\}$, where $\varepsilon_1 < \min\{\varepsilon_0, 2\varkappa\}$. Let $\hat{V}(g_1) = V(g_1)$ be the pre-image of the set $V(0)$ under the change of variables $x \mapsto x'$. Assume that $\text{supp } u \subset V(g_1) \cap \overline{Q}$.

Introduce the function $v(x') = u(x(x'))$. Denote $x' = (y', z')$ by $x = (y_1, y_2, z)$. Then the boundary-value problem (6.9), (6.10) takes the form

$$-\frac{\partial^2 v}{\partial y_1^2} - \frac{\partial^2 v}{\partial y_2^2} - \frac{1}{(1-y_1)^2} \frac{\partial^2 v}{\partial z^2} - \frac{1}{1-y_1} \frac{\partial v}{\partial y_1} = f'_0(x), \quad x \in \Theta, \quad (6.11)$$

$$v(y, z)|_{\Gamma_{1l}} - \alpha_l v(\mathcal{G}_l y, z)|_{\Gamma_{1l}} = f'_l(x), \quad x \in \Gamma_{1l}, \quad l = 1, 2, \quad (6.12)$$

where \mathcal{G}_l is the operator of rotation by the angle $(-1)^{l+1}\pi/4$,

$$\Theta = \{x : r > 0, |\varphi| < \pi/4, z \in \mathbb{R}\}, \quad \Gamma_{1l} = \{x : r > 0, \varphi = (-1)^l \pi/4, z \in \mathbb{R}\},$$

and r, φ are the polar coordinates of the point y .

Clearly, Conditions 1.1–1.4 are fulfilled in this example.

Passing to the principal homogeneous part in Eq. (6.11) with the coefficients freezed at the origin, we obtain

$$-\Delta v = f'_0(x), \quad x \in \Theta. \quad (6.13)$$

Nonlocal boundary conditions (6.12) do not change.

In the case of problem (6.13), (6.12), the operator $\mathcal{L}_g = \mathcal{L} : H_a^2(\Theta) \rightarrow \mathcal{H}_a^0(\Theta, \Gamma)$ given by (1.11) takes the form

$$\mathcal{L}v = (-\Delta v, v(\varphi, r, z)|_{\Gamma_{11}} - \alpha_1 v(\varphi + \pi/4, r, z)|_{\Gamma_{11}}, v(\varphi, r, z)|_{\Gamma_{12}} - \alpha_2 v(\varphi - \pi/4, r, z)|_{\Gamma_{12}}). \quad (6.14)$$

Hence, the operators

$$\mathcal{L}_g(\omega) = \mathcal{L}(\omega) : E_a^2(\theta) \rightarrow \mathcal{E}_a^0(\theta, \gamma)$$

and

$$\hat{\mathcal{L}}_g(\lambda) = \hat{\mathcal{L}}(\lambda) : W^2(-\pi/4, \pi/4) \rightarrow \mathcal{W}^0[-\pi/4, \pi/4]$$

given by (2.1) and (2.2) have the form

$$\begin{aligned} \mathcal{L}(\omega)V &= (-\Delta_y V + V, \\ &V(\varphi, r)|_{\gamma_{11}} - \alpha_1 V(\varphi + \pi/4, r)|_{\gamma_{11}}, V(\varphi, r)|_{\gamma_{12}} - \alpha_2 V(\varphi - \pi/4, r)|_{\gamma_{12}}) \end{aligned} \quad (6.15)$$

and

$$\hat{\mathcal{L}}(\lambda)w = (-w_{\varphi\varphi} + \lambda^2 w, w(-\pi/4) - \alpha_1 w(0), w(\pi/4) - \alpha_2 w(0)), \quad (6.16)$$

respectively, where

$$\theta = \{y \in \mathbb{R}^2 : r > 0, |\varphi| < \pi/4\}, \quad \gamma_{1l} = \{y \in \mathbb{R}^2 : r > 0, \varphi = (-1)^l \pi/4\},$$

and $\omega = \pm 1$ (cf. Example 2.1 for $d = \pi/2$).

It follows from Example 2.1 that the strip $-1 \leq \text{Im } \lambda \leq 1$ contains no eigenvalues of the operator-valued function $\hat{\mathcal{L}}(\lambda)$ and the operator $\mathcal{L}(\omega)$, $\omega = \pm 1$, is an isomorphism for $0 \leq a \leq 2$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $0 < |\alpha_1 + \alpha_2| < 2$ and $\pi/4 < \arctan \sqrt{4(\alpha_1 + \alpha_2)^{-2} - 1}$. Therefore, by Theorem 2.2, the operator \mathcal{L} is an isomorphism for the above a and α_l .

Suppose that $a > 1$ and prove that the operators B_l^2 satisfy Condition 6.2 with $\varkappa_1 = 2\varkappa$, $\varkappa_2 = \varkappa$, and $\sigma = \varkappa$. Using Lemma 4.5, we have

$$\begin{aligned} \|B_l^2 u\|_{H_a^{3/2}(\Gamma_l)} &\leq k_1 \|F^{-1}(b_l F(\eta u))\|_{W^2(\Gamma_l \times (-\infty, \infty))} \\ &= \frac{k_1}{\sqrt{2\pi}} \sum_{|\alpha|+|\beta| \leq 2} \left\| D_y^\alpha D_t^\beta \int_{\mathbb{R}} e^{i\lambda t} b_l(\lambda) F(\eta u)(y, \lambda) d\lambda \right\|_{L_2(\Gamma_l \times (-\infty, \infty))} \\ &= \frac{k_1}{\sqrt{2\pi}} \sum_{|\alpha|+|\beta| \leq 2} \left\| \int_{\mathbb{R}} e^{i\lambda t} \lambda^\beta b_l(\lambda) D_y^\alpha F(\eta u)(y, \lambda) d\lambda \right\|_{L_2(\Gamma_l \times (-\infty, \infty))}, \end{aligned}$$

where $D_t = -i\partial/\partial t$. Applying the Plancherel theorem and using property (6.8), we obtain

$$\begin{aligned} \|B_l^2 u\|_{H_a^{3/2}(\Gamma_l)} &\leq \frac{k_1}{\sqrt{2\pi}} \sum_{|\alpha|+|\beta| \leq 2} \|\lambda^\beta b_l(\lambda) D_y^\alpha F(\eta u)(y, \lambda)\|_{L_2(\Gamma_l \times (-\infty, \infty))} \\ &\leq k_2 \sum_{|\alpha|+|\beta| \leq 2} \|\lambda^\beta D_y^\alpha F(\eta u)(y, \lambda)\|_{L_2(\Gamma_l \times (-\infty, \infty))}. \end{aligned}$$

Applying the Plancherel theorem once more and taking into account that $\text{supp } \eta \subset (2\varkappa, 5\varkappa)$, we have

$$\|B_l^2 u\|_{H_a^{3/2}(\Gamma_l)} \leq k_3 \|u\|_{W^2(\Omega_{l\varkappa}^1)},$$

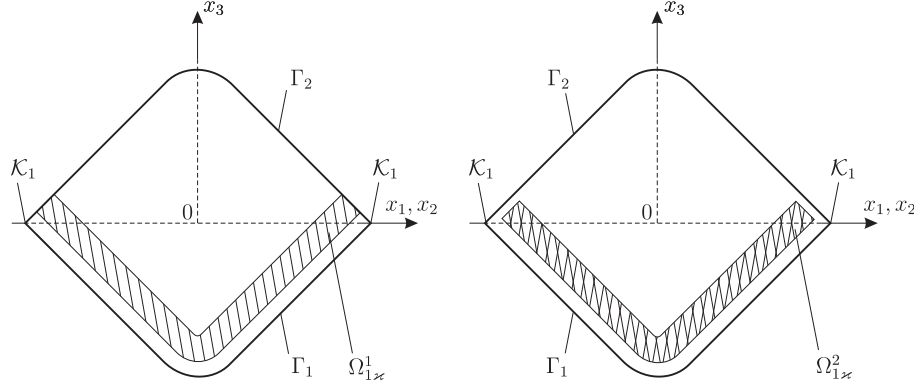


Figure 6.1: Problem (6.9), (6.10)

where

$$\Omega_{l\kappa}^1 = \{x = y + n_y t : y \in \Gamma_l, t \in (2\kappa, 5\kappa)\}$$

(see Fig. 6.1). Since $\Omega_{l\kappa}^1 \subset Q \setminus \overline{\mathcal{K}_1^{2\kappa}}$ and the norms in $W^2(Q \setminus \overline{\mathcal{K}_1^{2\kappa}})$ and $H_a^2(Q \setminus \overline{\mathcal{K}_1^{2\kappa}})$ are equivalent, the last inequality yields

$$\|B_l^2 u\|_{H_a^{3/2}(\Gamma_l)} \leq k_4 \|u\|_{H_a^2(Q \setminus \overline{\mathcal{K}_1^{2\kappa}})}. \quad (6.17)$$

Denote

$$\Omega_{l\kappa}^2 = \{x = y + n_y t : y \in \Gamma_l \setminus \overline{\mathcal{K}_1^{2\kappa}}, t \in (2\kappa, 5\kappa)\}.$$

Since $\Omega_{l\kappa}^2 \subset Q_\kappa$, similarly to (6.17), we obtain

$$\|B_l^2 u\|_{H_a^{3/2}(\Gamma_l \setminus \overline{\mathcal{K}_1^{2\kappa}})} \leq k_5 \|u\|_{H_a^2(Q_\kappa)}. \quad (6.18)$$

It follows from (6.17) and (6.18) that the operators B_l^2 satisfy Condition 6.2 for $a > 1$.

We consider the linear bounded operators

$$\mathbf{L}, \mathbf{L}^1 : H_a^2(Q) \rightarrow H_a^0(Q) \times H_a^{3/2}(\Gamma_1) \times H_a^{3/2}(\Gamma_2)$$

given by

$$\mathbf{L}u = \{-\Delta u, u|_{\Gamma_l} + B_l^1 u + B_l^2 u\}, \quad \mathbf{L}^1 u = \{-\Delta u, u|_{\Gamma_l} + B_l^1 u\}.$$

It follows from Theorem 5.1 that the operator \mathbf{L}^1 has the Fredholm property for $0 \leq a \leq 2$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $0 < |\alpha_1 + \alpha_2| < 2$ and $\pi/4 < \arctan \sqrt{4(\alpha_1 + \alpha_2)^{-2} - 1}$. By Theorem 6.3, the operator \mathbf{L} has the Fredholm property and $\text{ind } \mathbf{L} = \text{ind } \mathbf{L}^1$ for $1 < a \leq 2$, $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $0 < |\alpha_1 + \alpha_2| < 2$ and $\pi/4 < \arctan \sqrt{4(\alpha_1 + \alpha_2)^{-2} - 1}$, and a continuous function $b(\lambda)$ satisfying relation (6.8).

7 Setting of Nonlocal Elliptic Problems with a Parameter. Model Operators

7.1

In Secs. 7 and 8, we prove the unique solvability of nonlocal elliptic problems with a parameter $p = (p_1, \dots, p_d) \in \mathbb{R}^d$, where $d \geq 1$. Similarly to the above, we first establish the unique solvability of model nonlocal problems with a parameter in dihedral angles. Combining these results with those in [2] and [20] and making use of a partition of unity, we will consider nonlocal problems in bounded domains.

Let the domain Q , the transformations ω_{is} , and the sets $\mathcal{K}_1, \mathcal{K}_2$, and K be the same as in Sec. 6.

To consider nonlocal problems with a parameter, we introduce norms for the weighted spaces, depending on this parameter. First, we introduce the norms on the dihedral angle

$$\Theta = \{x = (y, z) \in \mathbb{R}^n : d_1 < \varphi < d_2, z \in \mathbb{R}^{n-2}\} \quad (7.1)$$

and on the half-plane

$$\Gamma = \{x = (y, z) \in \mathbb{R}^n : \varphi = d', z \in \mathbb{R}^{n-2}\}, \quad d_1 \leq d' \leq d_2. \quad (7.2)$$

Consider the space $V_a^k(\Theta) = H_a^k(\Theta) \cap H_a^0(\Theta)$ with the norm

$$\|u\|_{V_a^k(\Theta)} = (\|u\|_{H_a^k(\Theta)}^2 + |p|^{2k} \|u\|_{H_a^0(\Theta)}^2)^{1/2} \quad (u \in V_a^k(\Theta)), \quad (7.3)$$

where $k \geq 0$ is an integer.

For an integer $\nu \geq 0$, we denote

$$\|\psi\|_{H_a^\nu(\Gamma)}^2 = \sum_{|\alpha| \leq \nu} \int_{\Gamma} r^{2(a-\nu+|\alpha|)} |D^\alpha \psi|^2 d\Gamma.$$

Consider the space $V_a^{k-1/2}(\Gamma) = H_a^{k-1/2}(\Gamma) \cap H_a^0(\Gamma)$ with the norm

$$\|\psi\|_{V_a^{k-1/2}(\Gamma)} = (\|\psi\|_{H_a^{k-1/2}(\Gamma)}^2 + |p|^{2(k-1/2)} \|\psi\|_{H_a^0(\Gamma)}^2)^{1/2} \quad (\psi \in V_a^{k-1/2}(\Gamma), k \geq 1). \quad (7.4)$$

Now we introduce the norms for the domain Q and for the manifolds Γ_i . Set²

$$\|u\|_{H_a^k(Q)} = (\|u\|_{H_a^k(Q)}^2 + |p|^{2k} \|u\|_{H_a^0(Q)}^2)^{1/2} \quad (u \in H_a^k(Q)),$$

$$\|\psi\|_{H_a^{k-1/2}(\Gamma_i)} = (\|\psi\|_{H_a^{k-1/2}(\Gamma_i)}^2 + |p|^{2(k-1/2)} \|\psi\|_{H_a^0(\Gamma_i)}^2)^{1/2} \quad (\psi \in H_a^{k-1/2}(\Gamma_i), k \geq 1),$$

where

$$\|\psi\|_{H_a^0(\Gamma_i)}^2 = \int_{\Gamma_i} \rho^{2a} |\psi|^2 d\Gamma_i$$

and $\rho(x)$ is the function occurring in the definition of the spaces $H_a^k(Q)$.

We also set

$$\|f\|_{\mathcal{H}_a^l(Q, \Gamma)} = \left(\|f_0\|_{H_a^l(Q)}^2 + \sum_{i, \mu} \|f_{i\mu}\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)}^2 \right)^{1/2}$$

for $f = \{f_0, f_{i\mu}\} \in \mathcal{H}_a^l(Q, \Gamma)$.

Lemma 7.1. 1. For all $u \in V_a^k(\Theta)$, $k \geq 2$ is an integer, $p \in \mathbb{R}^d$, and integer s , $0 < s < k$, we have

$$|p|^{k-s} \|u\|_{H_a^s(\Theta)} \leq c_1 \|u\|_{V_a^k(\Theta)}, \quad (7.5)$$

where Θ is defined in (7.1) and $c_1 = c_1(k, s) > 0$ does not depend on u and p .

2. For all $u \in H_a^k(Q)$, $k \geq 2$ is an integer, $p \in \mathbb{R}^d$, and integer s , $0 < s < k$, we have

$$|p|^{k-s} \|u\|_{H_a^s(Q)} \leq c_2 \|u\|_{H_a^k(Q)}, \quad (7.6)$$

where $c_2 = c_2(k, s) > 0$ does not depend on u and p .

Proof. First, we prove the interpolation inequality (7.5).

Let $\{\xi_l\}_{l=-\infty}^{+\infty}$ be a partition of unity subordinated to the covering of the angle $\theta = \{y \in \mathbb{R}^2 : d_1 < \varphi < d_2\}$ by the sets $\theta_l = \{y \in \theta : 2^{l-1} < r < 2^{l+1}\}$ such that

$$|D^\alpha \xi_l(y)| \leq k_\alpha 2^{-|\alpha|l}, \quad y \in \theta_l, \quad l = 0, \pm 1, \pm 2, \dots, \quad (7.7)$$

where $k_\alpha > 0$ does not depend on l .

Denote $\Theta_l = \theta_l \times \mathbb{R}^{n-2}$, $l = 0, \pm 1, \pm 2, \dots$

Using (7.7) and the fact that $\text{supp } \xi_l \cap \overline{\Theta_j} = \emptyset$ for $j \neq l-1, l, l+1$, one can easily verify that

$$\|u\|_{H_a^s(\Theta)} \approx \left(\sum_{l=-\infty}^{+\infty} \|\xi_l u\|_{H_a^s(\Theta_l)}^2 \right)^{1/2} \quad (7.8)$$

for $s = 0, 1, 2, \dots$, where the symbol \approx means the equivalence of the norms.

Using the theorem about extension of functions from a domain with Lipschitz boundary to \mathbb{R}^n and applying the interpolation inequality for the Sobolev space $W^k(\mathbb{R}^n)$ (see [2, Sec. 1]), we obtain

$$q^{2(k-s)} \|v\|_{W^s(\Theta_0)}^2 \leq k_1 (\|v\|_{W^k(\Theta_0)}^2 + q^{2k} \|v\|_{L_2(\Theta_0)}^2) \quad (7.9)$$

for all $v \in W^k(\Theta_0)$ and $q > 0$, where $k_1 > 0$ does not depend on v and q .

²We do not introduce the spaces $V_a^k(Q) = H_a^k(Q) \cap H_a^0(Q)$ and $V_a^{k-1/2}(\Gamma_i) = H_a^{k-1/2}(\Gamma_i) \cap H_a^0(\Gamma_i)$ because they coincide with $H_a^k(Q)$ and $H_a^{k-1/2}(\Gamma_i)$, respectively, in the case of bounded domain Q .

We introduce the new variables $x' = 2^{-l}x$. Using the equivalence of the norms (7.8) and interpolation inequality (7.9) with $q = |p|2^l$ and passing back to the variables $x = 2^l x'$, we have

$$\begin{aligned} |p|^{2(k-s)} \|u\|_{H_a^s(\Theta)}^2 &\leq k_2 |p|^{2(k-s)} \sum_{l=-\infty}^{+\infty} 2^{(2(a-s)+n)l} \sum_{|\alpha| \leq s} \int_{\Theta_0} |D_{x'}^\alpha (\xi_l u)(x')|^2 dx' \\ &\leq k_3 \sum_{l=-\infty}^{+\infty} 2^{(2(a-k)+n)l} \left(\sum_{|\alpha| \leq k} \int_{\Theta_0} |D_{x'}^\alpha (\xi_l u)(x')|^2 dx' + |p|^{2k} 2^{2lk} \int_{\Theta_0} |(\xi_l u)(x')|^2 dx' \right) \\ &\leq k_4 \sum_{l=-\infty}^{+\infty} (\|\xi_l u\|_{H_a^k(\Theta_l)}^2 + |p|^{2k} \|\xi_l u\|_{H_a^0(\Theta_l)}^2), \end{aligned}$$

where $k_2, k_3, k_4 > 0$ do not depend on u and p . Combining this inequality with (7.8) yields (7.5).

2. Note that the relation $u \in H_a^k(\Theta)$ implies that $u \in V_a^k(\Theta)$, provided that u is compactly supported. Therefore, using a partition of unity, interpolation inequality (7.5), and the interpolation inequality of the kind (7.9) for Sobolev spaces, we obtain (7.6) for all $u \in H_a^k(Q)$ and $p \in \mathbb{R}^d$. \square

Lemma 7.2. 1. For all $u \in V_a^1(\Theta)$ and $p \in \mathbb{R}^d$, we have

$$|p|^{1/2} \|u|_\Gamma\|_{H_a^0(\Gamma)} \leq c_1 \|u\|_{V_a^1(\Theta)}, \quad (7.10)$$

where Θ and Γ are defined in (7.1) and (7.2) respectively, while $c_1 > 0$ does not depend on u and p .

2. For all $u \in H_a^1(Q)$ and $p \in \mathbb{R}^d$, we have

$$|p|^{1/2} \|u|_{\Gamma_i}\|_{H_a^0(\Gamma_i)} \leq c_2 \|u\|_{H_a^1(Q)}, \quad (7.11)$$

where $c_2 > 0$ does not depend on u and p .

Proof. Similarly to the proof of Lemma 7.1, it suffices to prove inequality (7.10).

Denote $\Gamma_l = \{x = (y, z) \in \Gamma : 2^{l-1} < r < 2^{l+1}\}$, $l = 0, \pm 1, \pm 2, \dots$. Let ξ_l be the same functions as in the proof of Lemma 7.1.

Similarly to (7.8), we have

$$\|u|_\Gamma\|_{H_a^0(\Gamma)}^2 \approx \sum_{l=-\infty}^{+\infty} \|(\xi_l u)|_{\Gamma_l}\|_{H_a^0(\Gamma_l)}^2, \quad (7.12)$$

where the symbol \approx means the equivalence of the norms.

Using the theorem about extension of functions from a domain with Lipschitz boundary to \mathbb{R}^n and applying the interpolation inequality for the Sobolev space $W^1(\mathbb{R}^n)$ (see [2, Sec. 1]), we obtain

$$q \|v|_{\Gamma_0}\|_{L_2(\Theta_0)}^2 \leq k_1 (\|v\|_{W^1(\Theta_0)}^2 + q^2 \|v\|_{L_2(\Theta_0)}^2), \quad (7.13)$$

for all $v \in W^k(\Theta_0)$ and $q > 0$, where $k_1 > 0$ does not depend on v and q .

We introduce the new variables $x' = 2^{-l}x$. Using the equivalence of the norms (7.12) and the interpolation inequality (7.13) with $q = |p|2^l$ and passing back to the variables $x = 2^l x'$, we have

$$\begin{aligned} |p| \cdot \|u|_\Gamma\|_{H_a^0(\Gamma)}^2 &\leq k_2 |p| \sum_{l=-\infty}^{+\infty} 2^{(2a+n-1)l} \int_{\Gamma_0} |(\xi_l u)(x')|_{\Gamma_0}|^2 d\Gamma_0 \\ &\leq k_3 \sum_{l=-\infty}^{+\infty} 2^{(2a+n-2)l} \left\{ \sum_{|\alpha| \leq 1} \int_{\Theta_0} |D_{x'}^\alpha (\xi_l u)(x')|^2 dx' + |p|^2 2^{2l} \int_{\Theta_0} |(\xi_l u)(x')|^2 dx' \right\} \\ &\leq k_4 \sum_{l=-\infty}^{+\infty} (\|\xi_l u\|_{H_a^1(\Theta_l)}^2 + |p|^2 \|\xi_l u\|_{H_a^0(\Theta_l)}^2), \end{aligned}$$

where $k_2, k_3, k_4 > 0$ do not depend on u and p . Combining this inequality with (7.8) yields (7.10). \square

In particular, it follows from Lemmas 7.1 and 7.2 that

$$\|u|_\Gamma\|_{V_a^{k-1/2}(\Gamma)} \leq c \|u\|_{V_a^k(\Theta)}, \quad (7.14)$$

$$\|u|_{\Gamma_i}\|_{H_a^{k-1/2}(\Gamma_i)} \leq C \|u\|_{H_a^k(Q)}, \quad (7.15)$$

where $c, C > 0$ do not depend on u and p .

Lemma 7.3. For all $\psi \in V_a^{k-1/2}(\Gamma)$, $k \geq 2$ is an integer, $p \in \mathbb{R}^d$, and integer s , $0 < s < k$, we have

$$|p|^{k-s-1/2} \|\psi\|_{H_a^s(\Gamma)} \leq c \|\psi\|_{V_a^{k-1/2}(\Gamma)}, \quad (7.16)$$

where Γ is defined by (7.2) and $c > 0$ does not depend on ψ and p .

Proof. The proof is based on the following interpolation inequality for Sobolev spaces in \mathbb{R}^{n-1} (see [2, Sec. 1]):

$$q^{2(k-s-1/2)} \|v\|_{W^s(\mathbb{R}^{n-1})}^2 \leq k_1 (\|v\|_{W^{k-1/2}(\mathbb{R}^{n-1})}^2 + q^{2(k-1/2)} \|v\|_{L_2(\mathbb{R}^{n-1})}^2) \quad (7.17)$$

for all $v \in W^{k-1/2}(\mathbb{R}^{n-1})$, $q > 0$, and integer s , $0 < s < k$, where $k_1 > 0$ does not depend on v and q . We will use the following equivalent norm in the space $W^{k-1/2}(\mathbb{R}^{n-1})$:

$$\left(\sum_{|\alpha|=k-1} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |D^\alpha v(x_1) - D^\alpha v(x_2)|^2 \frac{dx_1 dx_2}{|x_1 - x_2|^n} + \sum_{|\alpha| \leq k-1} \int_{\mathbb{R}^{n-1}} |D^\alpha v(x)|^2 dx \right)^{1/2} \quad (7.18)$$

(see, e.g., [30]).

Denote $\Gamma_l = \{x = (y, z) \in \Gamma : 2^{l-1} < r < 2^{l+1}\}$, $l = 0, \pm 1, \pm 2, \dots$. Let ξ_l be the same functions as in the proof of Lemmas 7.1 and 7.2.

Using relations (7.7) and the fact that $\text{supp } \xi_l \cap \overline{\Theta_j} = \emptyset$ for $j \neq l-1, l, l+1$, one can easily verify that

$$\|\psi\|_{H_a^s(\Gamma)} \approx \left(\sum_{l=-\infty}^{+\infty} \|\xi_l \psi\|_{H_a^s(\Gamma_l)}^2 \right)^{1/2} \quad (7.19)$$

for $s = 0, 1/2, 1, 3/2, \dots$, where the symbol \approx means the equivalence of the norms (in particular, see Lemma 1.1 in [20] for noninteger s).

Further, introducing the new variables $x' = 2^{-l}x$ and using equivalence (7.19), the interpolation inequality (7.17) with $q = |p|2^l$, and the equivalent norm (7.18) in the Sobolev space (which is possible because the functions ξ_l are compactly supported), we obtain

$$\begin{aligned} & |p|^{2(k-s-1/2)} \|\psi\|_{H_a^s(\Gamma)}^2 \\ & \leq k_2 \sum_{l=-\infty}^{+\infty} 2^{2l(a-k+1/2)} 2^{l(n-1)} \cdot |p|^{2(k-s-1/2)} 2^{2l(k-s-1/2)} \sum_{|\alpha| \leq s} \int_{\Gamma_0} |D_{x'}^\alpha (\xi_l \psi)(x')|^2 dx' \\ & \leq k_3 \sum_{l=-\infty}^{+\infty} 2^{2l(a-k+1/2)} 2^{l(n-1)} \\ & \quad \times \left(\sum_{|\alpha|=k-1} \int_{\Gamma} \int_{\Gamma} |D_{x'}^\alpha (\xi_l \psi)(x'_1) - D_{x'}^\alpha (\xi_l \psi)(x'_2)|^2 \frac{dx'_1 dx'_2}{|x'_1 - x'_2|^n} \right. \\ & \quad \left. + \sum_{|\alpha| \leq k-1} \int_{\Gamma_0} |D_{x'}^\alpha (\xi_l \psi)(x')|^2 dx' + |p|^{2(k-1/2)} 2^{2l(k-1/2)} \int_{\Gamma_0} |(\xi_l \psi)(x')|^2 dx' \right) \\ & \leq k_4 \sum_{l=-\infty}^{+\infty} 2^{2l(a-k+1/2)} 2^{l(n-1)} \\ & \quad \times \left(\sum_{|\alpha|=k-1} \int_{\Gamma} \int_{\Gamma} ||y'_1|^a D_{x'}^\alpha (\xi_l \psi)(x'_1) - |y'_2|^a D_{x'}^\alpha (\xi_l \psi)(x'_2)|^2 \frac{dx'_1 dx'_2}{|x'_1 - x'_2|^n} \right. \\ & \quad \left. + \sum_{|\alpha| \leq k-1} \int_{\Gamma_0} |y'|^{2(a+|\alpha|-k+1/2)} |D_{x'}^\alpha (\xi_l \psi)(x')|^2 dx' \right. \\ & \quad \left. + |p|^{2(k-1/2)} 2^{2l(k-1/2)} \int_{\Gamma_0} |y'|^{2a} |(\xi_l \psi)(x')|^2 dx' \right), \end{aligned}$$

where $k_2, k_3, \dots > 0$ do not depend on ψ and p , $x'_i = (y'_i, z'_i) \in \Gamma$, $y'_i \in \mathbb{R}^2$, and $z'_i \in \mathbb{R}^{n-2}$, $i = 1, 2$. In the last inequality, we have also used the fact that $\text{supp } \xi_l \cap \overline{Q_j} = \emptyset$ for $j \neq l-1, l, l+1$. Passing

back to the variables $x = 2^l x'$, we have

$$\begin{aligned} & |p|^{2(k-s-1/2)} \|\psi\|_{H_a^s(\Gamma)}^2 \\ & \leq k_5 \sum_{l=-\infty}^{+\infty} \left(\sum_{|\alpha|=k-1} \int_{\Gamma} \int_{\Gamma} |y_1|^a D^\alpha(\xi_l \psi)(x_1) - |y_2|^a D^\alpha(\xi_l \psi)(x_2)|^2 \frac{dx_1 dx_2}{|x_1 - x_2|^n} \right. \\ & \quad \left. + \sum_{|\alpha| \leq k-1} \int_{\Gamma_l} |y|^{2(a+|\alpha|-k+1/2)} |D^\alpha(\xi_l \psi)(x)|^2 dx + |p|^{2(k-1/2)} \int_{\Gamma_l} |y|^{2a} |(\xi_l \psi)(x)|^2 dx \right), \end{aligned}$$

where $x_i = (y_i, z_i) \in \Gamma$, $y_i \in \mathbb{R}^2$, and $z_i \in \mathbb{R}^{n-2}$, $i = 1, 2$. It follows from this inequality and from Lemma 1.3 in [20] (about the equivalent norms in the weighted trace spaces) that

$$|p|^{2(k-s-1/2)} \|\psi\|_{H_a^s(\Gamma)}^2 \leq k_6 \sum_{l=-\infty}^{+\infty} (\|\xi_l \psi\|_{H_a^{k-1/2}(\Gamma_l)}^2 + |p|^{2(k-1/2)} \|\xi_l \psi\|_{H_a^0(\Gamma_l)}^2). \quad (7.20)$$

Combining (7.20) with the equivalence of the norms (7.19), we obtain (7.16). \square

7.2

Consider the differential operators

$$\begin{aligned} A^0(p) &\equiv A^0(x, D, p) = \sum_{|\alpha|+|\beta|=2m} a_{\alpha\beta}(x) p^\beta D^\alpha, \\ B_{i\mu}^0(p) &\equiv B_{i\mu}^0(x, D, p) = \sum_{|\alpha|+|\beta|=m_{i\mu}} b_{i\mu s \alpha \beta}(x) p^\beta D^\alpha, \end{aligned}$$

where $a_{\alpha\beta}, b_{i\mu s \alpha \beta} \in C^\infty(\mathbb{R}^n)$ are complex-valued functions ($i = 1, \dots, N_0$; $\mu = 1, \dots, m$; $s = 0, \dots, S_i$), $\beta = (\beta_1, \dots, \beta_d)$, $|\beta| = |\beta_1| + \dots + |\beta_d|$, $p^\beta = p_1^{\beta_1} \dots p_d^{\beta_d}$, and $m_{i\mu} \leq 2m - 1$.

We study the following nonlocal elliptic problem:

$$A(p)u \equiv A^0(p)u + A^1(p)u = f_0(x), \quad x \in Q, \quad (7.21)$$

$$B_{i\mu}(p)u \equiv \sum_{j=0}^3 B_{i\mu}^j(p)u = f_{i\mu}(x), \quad x \in \Gamma_i; \quad i = 1, \dots, N_0; \quad \mu = 1, \dots, m. \quad (7.22)$$

Here

$$B_{i\mu}^0(p)u = B_{i\mu 0}^0(p)u|_{\Gamma_i}, \quad B_{i\mu}^1(p)u = \sum_{s=1}^{S_i} (B_{i\mu s}^0(x, D, p)(\xi u))(\omega_{is}(x))|_{\Gamma_i},$$

the function ξ and the transformations ω_{is} are the same as in Sec. 6. In particular, we assume that Conditions 1.3 and 1.4 hold.

Introduce the variable $t = (t_1, \dots, t_d) \in \mathbb{R}^d$ and formally replace the expressions p^β in the operators $A^0(x, D_x, p)$ and $B_{i\mu 0}^0(x, D_x, p)$ by the differential operators D_t^β . Assume that the following conditions hold (cf. [2, 20]).

Condition 7.1. The operator $A^0(x, D_x, D_t)$ is properly elliptic for $(x, t) \in \overline{Q} \times \mathbb{R}^d$.

Condition 7.2. The system $\{B_{i\mu 0}^0(x, D_x, D_t)\}_{\mu=1}^m$ covers the operator $A^0(x, D_x, D_t)$ and is normal for all $i = 1, \dots, N_0$ and $(x, t) \in \overline{\Gamma}_i \times \mathbb{R}^d$.

We also assume that the following conditions for the operators $A^1(p)$, $B_{i\mu}^2(p)$, and $B_{i\mu}^3(p)$ hold.

Condition 7.3 (smallness of perturbations). We have

$$\|A^1(p)u\|_{H_a^l(Q)} \leq c_1 \|u\|_{H_a^{l+2m-1}(Q)},$$

$$\|B_{i\mu}^3(p)u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq c_2 \|u\|_{H_a^{l+2m-1}(Q)},$$

where $i = 1, \dots, N_0$, $\mu = 1, \dots, m$, and $c_1, c_2 > 0$ do not depend on u and p .

Condition 7.4 (separability from the conjugation points). There exist numbers $\sigma > 0$ and $\varkappa_1 > \varkappa_2 > 0$ such that

$$\|B_{i\mu}^2(p)u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq c_1 \|u\|_{H_a^{l+2m}(Q \setminus \overline{\mathcal{K}_1^{\varkappa_1}})} \quad (7.23)$$

for all $u \in H_a^{l+2m}(Q \setminus \overline{\mathcal{K}_1^{\varkappa_1}})$ and

$$\|B_{i\mu}^2(p)u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i \setminus \overline{\mathcal{K}_1^{\varkappa_2}})} \leq c_2 \|u\|_{H_a^{l+2m}(Q_\sigma)} \quad (7.24)$$

for all $u \in H_a^{l+2m}(Q_\sigma)$; here $i = 1, \dots, N_0$; $\mu = 1, \dots, m$; $c_1, c_2 > 0$ do not depend on u and p .

Remark 7.1. 1. It follows from Condition 7.3 and from Lemma 7.1 that

$$\|A^1(p)u\|_{H_a^l(Q)} \leq c_1 |p|^{-1} \|u\|_{H_a^{l+2m}(Q)},$$

$$\|B_{i\mu}^3(p)u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq c_2 |p|^{-1} \|u\|_{H_a^{l+2m}(Q)}.$$

Therefore, the norms of the operators $A^1(p)$ and $B_{i\mu}^3(p)$ are small, provided that $|p|$ is large.

2. Condition 7.4 is an analog of Condition 6.2.

Remark 7.2. Let the transformations ω_{is} and the set K be the same as in Secs. 1–5. Consider the problem

$$\begin{aligned} \sum_{|\alpha|+|\beta|\leq 2m} a_{\alpha\beta}(x) p^\beta D^\alpha u(x) &= f_0(x), \quad x \in Q, \\ \sum_{|\alpha|+|\beta|\leq m_{i\mu}} \sum_{s=0}^{S_i} b_{i\mu s \alpha\beta}(\omega_{is}(x)) p^\beta (D^\alpha u)(\omega_{is}(x))|_{\Gamma_i} &= f_{i\mu}(x), \\ x \in \Gamma_i; \quad i &= 1, \dots, N_0; \quad \mu = 1, \dots, m. \end{aligned}$$

This problem can be represented in the form (7.21), (7.22). Indeed, set

$$\begin{aligned} A^0(p) &= \sum_{|\alpha|+|\beta|=2m} a_{\alpha\beta}(x) p^\beta D^\alpha, \quad A^1(p) = \sum_{|\alpha|+|\beta|\leq 2m-1} a_{\alpha\beta}(x) p^\beta D^\alpha, \\ B_{i\mu}^0(p)u &= \sum_{|\alpha|+|\beta|=m_{i\mu}} b_{i\mu 0 \alpha\beta}(x) p^\beta D^\alpha u|_{\Gamma_i}, \\ B_{i\mu}^1(p)u &= \sum_{|\alpha|+|\beta|=m_{i\mu}} \sum_{s=1}^{S_i} b_{i\mu s \alpha\beta}(\omega_{is}(x)) p^\beta (D^\alpha(\xi u))(\omega_{is}(x))|_{\Gamma_i}, \\ B_{i\mu}^2(p)u &= \sum_{|\alpha|+|\beta|=m_{i\mu}} \sum_{s=1}^{S_i} b_{i\mu s \alpha\beta}(\omega_{is}(x)) p^\beta (D^\alpha((1-\xi)u))(\omega_{is}(x))|_{\Gamma_i}, \\ B_{i\mu}^3(p)u &= \sum_{|\alpha|+|\beta|\leq m_{i\mu}-1} \sum_{s=0}^{S_i} b_{i\mu s \alpha\beta}(\omega_{is}(x)) p^\beta (D^\alpha u)(\omega_{is}(x))|_{\Gamma_i}. \end{aligned}$$

Clearly, the operator $A^1(p)$ satisfies Condition 7.3.

Let $s = 0, \dots, S_i$ and $|\alpha|+|\beta| \leq m_{i\mu}-1$. Denote by u_1 an extension of the function u to $Q \cup \omega_{is}(\Omega_i)$, defined by Lemma 4.4 and satisfying the inequalities

$$\|u_1\|_{H_a^\nu(Q \cup \omega_{is}(\Omega_i))} \leq k_1 \|u\|_{H_a^\nu(Q)}, \quad \nu = 0, \dots, l+2m-1. \quad (7.25)$$

Clearly,

$$b_{i\mu s \alpha\beta}(\omega_{is}(x)) p^\beta (D^\alpha u)(\omega_{is}(x))|_{\Gamma_i} = b_{i\mu s \alpha\beta}(\omega_{is}(x)) p^\beta (D^\alpha u_1)(\omega_{is}(x))|_{\Gamma_i}.$$

Therefore, using (7.15) and (7.25), we have

$$\begin{aligned} \|b_{i\mu s \alpha\beta}(\omega_{is}(x)) p^\beta (D^\alpha u)(\omega_{is}(x))|_{\Gamma_i}\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \\ \leq k_2 \|u_1\|_{H_a^{l+2m-1}(\omega_{is}(\Omega_i))} \leq k_3 \|u\|_{H_a^{l+2m-1}(Q)} \end{aligned}$$

for α and β such that $|\alpha|+|\beta| \leq m_{i\mu}-1$. This inequality proves that the operators $B_{i\mu}^3(p)$ satisfy Condition 7.3.

To show that the operators $B_{i\mu}^2(p)$ satisfy Condition 7.4, one must repeat the proof of Lemma 4.6 with the norms $\|\cdot\|$ replaced by the norms $\|\cdot\|$, taking into account inequality (7.15).

We consider the linear bounded operators $\mathbf{L}^0(p), \mathbf{L}^1(p), \mathbf{L}(p) : H_a^{l+2m}(Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ given by

$$\mathbf{L}^0(p) = \{A(p), B_{i\mu}^0(p)\}, \quad \mathbf{L}^1(p) = \{A(p), B_{i\mu}^0(p) + B_{i\mu}^1(p)\}, \quad \mathbf{L}(p) = \{A(p), B_{i\mu}(p)\}.$$

The invertibility of the operator $\mathbf{L}^0(p)$ was proved in [20]. Our aim is to study the operator $\mathbf{L}^1(p)$ and then $\mathbf{L}(p)$. First, we will consider model problems with a parameter corresponding to the points of the sets \mathcal{K}_1 and \mathcal{K}_2 .

7.3

Fix a point $g \in \mathcal{K}_1$. Using the reasoning similar to that in Sec. 1, we arrive at the following model problem (cf. (1.9), (1.10)):

$$A_j(x, D_y, D_z, p)v_j(x) = f_j(x), \quad x \in \Theta_j; \quad j = 1, \dots, N, \quad (7.26)$$

$$\sum_{k=1}^N \sum_{s=0}^{S_{j\rho k}} (B_{j\rho\mu ks}(x, D_y, D_z, p)v_k)(\mathcal{G}_{j\rho ks}y, z)|_{\Gamma_{j\rho}} = f_{j\rho\mu}(x), \quad x \in \Gamma_{j\rho}; \quad (7.27)$$

$$j = 1, \dots, N; \quad \rho = 1, 2; \quad \mu = 1, \dots, m,$$

where $A_j(x, D_y, D_z, p)$ and $B_{j\rho\mu ks}(x, D_y, D_z, p)$ are differential operators of order $2m$ and $m_{j\rho\mu}$, respectively, with the parameter p , having variable coefficients of class C^∞ , while Θ_j , $\Gamma_{j\rho}$, and $\mathcal{G}_{j\rho ks}$ are the same as in (1.9), (1.10).

Introduce the spaces of vector-valued functions

$$\mathcal{V}_a^k(\Theta) = \prod_{j=1}^N V_a^k(\Theta_j), \quad \mathcal{V}_a^l(\Theta, \Gamma) = \mathcal{V}_a^l(\Theta) \times \prod_{j=1}^N \prod_{\rho=1,2} \prod_{\mu=1}^m V_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho}),$$

where $m_{j\rho\mu}$ is the order of the operator $B_{j\rho\mu ks}(x, D_y, D_z, p)$. Consider the linear bounded operator $\mathcal{L}_g(p) : \mathcal{V}_a^{l+2m}(\Theta) \rightarrow \mathcal{V}_a^l(\Theta, \Gamma)$ given by

$$\mathcal{L}_g(p)v = \left\{ A_j(D_y, D_z, p)v_j(y, z), \sum_{k,s} (B_{j\rho\mu ks}(D_y, D_z, p)v_k)(\mathcal{G}_{j\rho ks}y, z)|_{\Gamma_{j\rho}} \right\}, \quad (7.28)$$

where $A_j(D_y, D_z, p)$ and $B_{j\rho\mu ks}(D_y, D_z, p)$ are principal homogeneous parts³ of the operators $A_j(0, D_y, D_z, p)$ and $B_{j\rho\mu ks}(0, D_y, D_z, p)$, respectively. Clearly, if we replace p^β by D_t^β , then each of the obtained operators $A_j(D_y, D_z, D_t)$ will be properly elliptic for $(x, t) \in \overline{\Theta_j} \times \mathbb{R}^d$, while the system $\{B_{j\rho\mu j0}(D_y, D_z, D_t)\}_{\mu=1}^m$ will cover the operator $A_j(D_y, D_z, D_t)$ and be normal for all $(x, t) \in \Gamma_{j\rho} \times \mathbb{R}^d$, $j = 1, \dots, N$, and $\rho = 1, 2$.

We also set

$$\mathcal{L}'_g(p)v = \left\{ A_j^0(x, D_y, D_z, p)v_j(y, z), \sum_{k,s} (B_{j\rho\mu ks}^0(x, D_y, D_z, p)v_k)(\mathcal{G}_{j\rho ks}y, z)|_{\Gamma_{j\rho}} \right\},$$

where $A_j^0(x, D_y, D_z, p)$ and $B_{j\rho\mu ks}^0(x, D_y, D_z, p)$ are principal homogeneous parts of the operators $A_j(x, D_y, D_z, p)$ and $B_{j\rho\mu ks}(x, D_y, D_z, p)$, respectively.

Further, we set

$$\mathcal{L}_g(\eta, p)V = \left\{ A_j(D_y, \eta, p)V_j(y), \sum_{k,s} (B_{j\rho\mu ks}(D_y, \eta, p)V_k)(\mathcal{G}_{j\rho ks}y)|_{\Gamma_{j\rho}} \right\}, \quad \eta \in \mathbb{R}^{n-2}.$$

Replacing (η, p) by $\omega = (\eta, p)/|(\eta, p)|$, we obtain the bounded operator

$$\mathcal{L}_g(\omega) : \mathcal{E}_a^{l+2m}(\theta) \rightarrow \mathcal{E}_a^l(\theta, \gamma)$$

given by

$$\mathcal{L}_g(\omega)V = \left\{ A_j(D_y, \omega)V_j(y), \sum_{k,s} (B_{j\rho\mu ks}(D_y, \omega)V_k)(\mathcal{G}_{j\rho ks}y)|_{\Gamma_{j\rho}} \right\}, \quad \omega \in S^{n+d-3}. \quad (7.29)$$

Finally, we consider the analytic operator-valued function

$$\hat{\mathcal{L}}_g(\lambda) : \mathcal{W}^{l+2m}(d_1, d_2) \rightarrow \mathcal{W}^l[d_1, d_2]$$

given by (2.2).

In this subsection, we prove that the absence of eigenvalues of $\hat{\mathcal{L}}_g(\lambda)$ on the line $\text{Im } \lambda = a+1-l-2m$ and the triviality of the kernel and cokernel of $\mathcal{L}_g(\omega)$ guarantees the existence of the inverse operators $\mathcal{L}_g^{-1}(p)$ for $p \in \mathbb{R}^d \setminus \{0\}$, uniformly bounded in the corresponding norms $\|\cdot\|$. We introduce these norms by setting

$$\|u\|_{\mathcal{V}_a^k(\Theta)} = \left(\sum_j \|u\|_{V_a^k(\Theta_j)}^2 \right)^{1/2},$$

$$\|f\|_{\mathcal{V}_a^l(\Theta, \Gamma)} = \left(\sum_j \|f_j\|_{V_a^l(\Theta_j)}^2 + \sum_{j, \rho, \mu} \|f_{j\rho\mu}\|_{V_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})}^2 \right)^{1/2}, \quad f = \{f_j, f_{j\rho\mu}\}.$$

³In this section, the notion “principal homogeneous part” takes into account the parameter p , e.g., the operator $A_j(D_y, D_z, p)$ consists of the terms $a_{j\alpha\beta}(x)p^\beta D^\alpha$, where $|\alpha| + |\beta| = 2m$.

Theorem 7.1. *Let Conditions 7.1, 7.2, 1.3, and 1.4 hold. Assume that the line $\text{Im } \lambda = a + 1 - l - 2m$ contains no eigenvalues of $\mathcal{L}_g(\lambda)$ and $\dim \mathcal{N}(\mathcal{L}_g(\omega)) = \text{codim } \mathcal{R}(\mathcal{L}_g(\omega)) = 0$ for any $\omega \in S^{n+d-3}$. Then the operator $\mathcal{L}_g(p)$ is an isomorphism for $p \in \mathbb{R}^d \setminus \{0\}$ and*

$$\|u\|_{\mathcal{V}_a^{l+2m}(\Theta)} \leq c \|\mathcal{L}_g(p)u\|_{\mathcal{V}_a^l(\Theta, \Gamma)}, \quad p \in \mathbb{R}^d \setminus \{0\}, \quad (7.30)$$

where $c > 0$ does not depend on u and p .

To prove Theorem 7.1, we preliminary consider the bounded operator

$$\mathcal{L}_g(p) : \mathcal{E}_a^{l+2m}(\Theta) \rightarrow \mathcal{E}_a^l(\Theta, \Gamma)$$

given by (7.28) for $p \in S^{d-1}$, where

$$\mathcal{E}_a^k(\Theta) = \prod_{j=1}^N E_a^k(\Theta_j), \quad \mathcal{E}_a^l(\Theta, \Gamma) = \mathcal{E}_a^l(\Theta) \times \prod_{j=1}^N \prod_{\rho=1,2} \prod_{\mu=1}^m E_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho}),$$

while $E_a^k(\Theta_j)$ is the completion of the set $C_0^\infty(\overline{\Theta_j} \setminus \{0\})$ with respect to the norm

$$\|v\|_{E_a^k(\Theta_j)} = \left(\sum_{|\alpha| \leq k} \int_{\Theta_j} |y|^{2a} (|y|^{2(|\alpha|-k)} + 1) |D_x^\alpha v(x)|^2 dx \right)^{1/2}$$

and $E_a^{k-1/2}(\Gamma_{j\rho})$ ($k \geq 1$ is an integer) is the space of traces on $\Gamma_{j\rho}$ with the norm

$$\|\psi\|_{E_a^{k-1/2}(\Gamma_{j\rho})} = \inf \|v\|_{E_a^k(\Theta_j)} \quad (v \in E_a^k(\Theta_j) : v|_{\Gamma_{j\rho}} = \psi).$$

Lemma 7.4. *Let Conditions 7.2, 1.3, and 1.4 hold, and let $f_{j\rho\mu} \in E_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})$. Then there exists a function $u \in \mathcal{E}_a^{l+2m}(\Theta)$ such that*

$$\sum_{k,s} (B_{j\rho\mu ks}(D_y, D_z, p)u_k)(\mathcal{G}_{j\rho ks}y, z)|_{\Gamma_{j\rho}} = f_{j\rho\mu}, \quad p \in S^{d-1},$$

$$\|u\|_{\mathcal{E}_a^{l+2m}(\Theta)} \leq c \sum_{j,\rho,\mu} \|f_{j\rho\mu}\|_{E_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})}, \quad p \in S^{d-1},$$

where $c > 0$ does not depend on u and p .

Proof. By Lemma 9.2' in [20], there exists a function $v = (v_1, \dots, v_N) \in \mathcal{E}_a^{l+2m}(\Theta)$ such that $B_{j\rho\mu j0}(D_y, D_z, p)v_j|_{\Gamma_{j\rho}} = f_{j\rho\mu}$ and

$$\|v\|_{\mathcal{E}_a^{l+2m}(\Theta)} \leq k_1 \sum_{j,\rho,\mu} \|f_{j\rho\mu}\|_{E_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})}, \quad p \in S^{d-1}.$$

Let d_0 be the number defined in (5.21). Consider functions $\xi_k \in C^\infty(\mathbb{R})$ depending on the polar angle φ of the point $y \in \mathbb{R}^2$, such that $\xi_k(\varphi) = 1$ for $d_{k1} \leq \varphi \leq d_{k1} + d_0/2$ and $d_{k2} - d_0/2 \leq \varphi \leq d_{k2}$ and $\xi_k(\varphi) = 0$ for $d_{k1} + d_0 \leq \varphi \leq d_{k2} - d_0$.

Since the operator of multiplication by ξ_j is bounded in $E_a^{l+2m}(\Theta_j)$, we see that the function $u = (\xi_1 v_1, \dots, \xi_N v_N)$ is the desired one. \square

Lemma 7.5. *Let the conditions of Theorem 7.1 be fulfilled. Then the operator $\mathcal{L}_g(p) : \mathcal{E}_a^{l+2m}(\Theta) \rightarrow \mathcal{E}_a^l(\Theta, \Gamma)$ is an isomorphism for $p \in S^{d-1}$ and*

$$\|u\|_{\mathcal{E}_a^{l+2m}(\Theta)} \leq c \|\mathcal{L}_g(p)u\|_{\mathcal{E}_a^l(\Theta, \Gamma)}, \quad p \in S^{d-1},$$

where $c > 0$ does not depend on u and p .

Proof. 1. Due to Lemma 7.4, it suffices to prove the unique solvability of the problem

$$A_j(D_y, D_z, p)u_j(y, z) = f_j(y, z), \quad (y, z) \in \Theta_j, \quad (7.31)$$

$$\sum_{k,s} (B_{j\rho\mu ks}(D_y, D_z, p)u_k)(\mathcal{G}_{j\rho ks}y, z)|_{\Gamma_{j\rho}} = 0 \quad (7.32)$$

and to show that

$$\|u\|_{\mathcal{E}_a^{l+2m}(\Theta)} \leq k_1 \|\{f_j\}\|_{\mathcal{E}_a^l(\Theta)}, \quad p \in S^{d-1},$$

where $k_1 > 0$ does not depend on u and p .

2. Making the Fourier transform with respect to z , we see that problem (7.31), (7.32) is equivalent to the following one:

$$A_j(D_y, \eta, p) \tilde{u}_j(y, \eta) = \tilde{f}_j(y, \eta), \quad y \in \theta_j, \quad \eta \in \mathbb{R}^{n-2}, \quad (7.33)$$

$$\sum_{k,s} (B_{j\rho\mu ks}(D_y, \eta, p) \tilde{u}_k)(\mathcal{G}_{j\rho ks} y, \eta)|_{\gamma_{j\rho}} = 0, \quad \eta \in \mathbb{R}^{n-2}, \quad (7.34)$$

where $\tilde{u}_j(y, \eta)$ is the Fourier transform of $u_j(y, z)$ with respect to z .

Set $\tilde{u}_j(y, \eta) = |(\eta, p)|^{-2m} U_j(|(\eta, p)|y, \eta)$ and $\tilde{f}_j(y, \eta) = F_j(|(\eta, p)|y, \eta)$. Then problem (7.33), (7.34) takes the form

$$A_j(D_Y, \omega) U_j(Y, \eta) = F_j(Y, \eta), \quad y \in \theta_j, \quad \eta \in \mathbb{R}^{n-2}, \quad (7.35)$$

$$\sum_{k,s} (B_{j\rho\mu ks}(D_Y, \omega) U_k)(\mathcal{G}_{j\rho ks} Y, \eta)|_{\gamma_{j\rho}} = 0, \quad \eta \in \mathbb{R}^{n-2}, \quad (7.36)$$

where $\omega = (\eta, p)/|(\eta, p)| \in S^{n+d-3}$ and $Y = |(\eta, p)|y$.

It follows from the conditions of the lemma and from Theorem 2.1 that problem (7.35), (7.36) has a unique solution $U \in \mathcal{E}_a^{l+2m}(\theta)$ for any right-hand side $\{F_j\} \in \mathcal{E}_a^l(\theta)$ and

$$\|U\|_{\mathcal{E}_a^{l+2m}(\theta)} \leq k_2 \|\{F_j\}\|_{\mathcal{E}_a^l(\theta)}, \quad \omega \in S^{n+d-3},$$

where $k_2 > 0$ does not depend on u and ω .

Thus, the lemma will be proved if we show that

$$\|\{f_j\}\|_{\mathcal{E}_a^l(\Theta)}^2 \approx \int_{\mathbb{R}^{n-2}} |(\eta, p)|^{-2(a-l+1)} \|\{F_j(\cdot, \eta)\}\|_{\mathcal{E}_a^l(\theta)}^2 d\eta, \quad (7.37)$$

$$\|u\|_{\mathcal{E}_a^{l+2m}(\Theta)}^2 \approx \int_{\mathbb{R}^{n-2}} |(\eta, p)|^{-2(a-l+1)} \|U(\cdot, \eta)\|_{\mathcal{E}_a^{l+2m}(\theta)}^2 d\eta; \quad (7.38)$$

here the symbol \approx between two expressions means that the first expression can be estimated from the below and from the above by the second expression with positive constants independent of $p \in S^{d-1}$.

3. Let us prove relation (7.37). Using the Parseval equality, we have

$$\begin{aligned} \|f_j\|_{E_a^l(\Theta_j)}^2 &= \sum_{|\alpha|+|\beta| \leq l} \int_{\theta_j} \int_{\mathbb{R}^{n-2}} |y|^{2a} (|y|^{2(|\alpha|+|\beta|-l)} + 1) |D_y^\alpha D_z^\beta f_j(y, z)|^2 dy dz \\ &= \sum_{|\alpha|+|\beta| \leq l} \int_{\theta_j} \int_{\mathbb{R}^{n-2}} |y|^{2a} (|y|^{2(|\alpha|+|\beta|-l)} + 1) |\eta^\beta|^2 |D_y^\alpha \tilde{f}_j(y, \eta)|^2 dy d\eta. \end{aligned}$$

Using Fubini's theorem and making the change of variables $Y = |(\eta, p)|y$ for each fixed η , we obtain

$$\begin{aligned} \|f_j\|_{E_a^l(\Theta_j)}^2 &= \sum_{|\alpha|+|\beta| \leq l} \int_{\mathbb{R}^{n-2}} \int_{\theta_j} |(\eta, p)|^{-2(a+1-|\alpha|)} |Y|^{2a} \\ &\quad \times (|(\eta, p)|^{-2(|\alpha|+|\beta|-l)} |Y|^{2(|\alpha|+|\beta|-l)} + 1) |\eta^\beta|^2 |D_Y^\alpha F(Y, \eta)|^2 dY d\eta \\ &= \sum_{|\alpha| \leq l} \sum_{\nu=0}^{l-|\alpha|} \int_{\mathbb{R}^{n-2}} \int_{\theta_j} \Phi_{\alpha\nu}(\eta, p, Y) |D_Y^\alpha F(Y, \eta)|^2 dY d\eta, \quad (7.39) \end{aligned}$$

where

$$\Phi_{\alpha\nu}(\eta, p, Y) = \sum_{|\beta|=\nu} |(\eta, p)|^{-2(a-l+1)} |Y|^{2a} \frac{|\eta^\beta|^2}{|(\eta, p)|^{2|\beta|}} (|Y|^{2(|\alpha|+|\beta|-l)} + |(\eta, p)|^{2(|\alpha|+|\beta|-l)}).$$

Note that $p \in S^{d-1}$ and $|\alpha| + |\beta| - l \leq 0$, which implies that

$$|(\eta, p)|^{2(|\alpha|+|\beta|-l)} \leq 1.$$

Therefore,

$$|Y|^{2(|\alpha|+|\beta|-l)} + |(\eta, p)|^{2(|\alpha|+|\beta|-l)} \leq |Y|^{2(|\alpha|+|\beta|-l)} + 1. \quad (7.40)$$

Furthermore, since $p \in S^{d-1}$, we have

$$\sum_{|\beta|=\nu} \frac{|\eta^\beta|^2}{|(\eta, p)|^{2|\beta|}} \leq k_3 \quad (\eta \in \mathbb{R}^{n-2}, p \in S^{d-1}, \nu = 0, \dots, l), \quad (7.41)$$

where $k_3 > 0$ does not depend on η and p .

Relations (7.39)–(7.41) imply that

$$\begin{aligned} & \|f_j\|_{E_a^l(\Theta_j)}^2 \\ & \leq k_3 \sum_{|\alpha| \leq l} \sum_{\nu=0}^{l-|\alpha|} \int_{\mathbb{R}^{n-2}} \int_{\Theta_j} |(\eta, p)|^{-2(a-l+1)} |Y|^{2a} (|Y|^{2(|\alpha|+\nu-l)} + 1) |D_Y^\alpha F_j(Y, \eta)|^2 dY d\eta. \end{aligned} \quad (7.42)$$

Clearly,

$$|Y|^{2(|\alpha|-l)} + 1 \leq \sum_{\nu=0}^{l-|\alpha|} |Y|^{2(|\alpha|+\nu-l)} \leq k_4 (|Y|^{2(|\alpha|-l)} + 1), \quad Y \in \mathbb{R}^2, \quad (7.43)$$

where $k_4 > 0$ does not depend on Y .

Inequalities (7.42) and (7.43) imply that

$$\|f_j\|_{E_a^l(\Theta_j)}^2 \leq k_3 k_4 \int_{\mathbb{R}^{n-2}} |(\eta, p)|^{-2(a-l+1)} \|F_j(\cdot, \eta)\|_{E_a^l(\Theta_j)}^2 d\eta. \quad (7.44)$$

Now we estimate the norm $\|f_j\|_{E_a^l(\Theta_j)}^2$ from below. To do this, we write it as follows:

$$\begin{aligned} \|f_j\|_{E_a^l(\Theta_j)}^2 &= \sum_{|\alpha| \leq l} \sum_{\nu=0}^{l-|\alpha|} \left(\int_{|\eta| < 1} \int_{\Theta_j} \Phi_{\alpha\nu}(\eta, p, Y) |D_Y^\alpha F(Y, \eta)|^2 dY d\eta \right. \\ &\quad \left. + \int_{|\eta| > 1} \int_{\Theta_j} \Phi_{\alpha\nu}(\eta, p, Y) |D_Y^\alpha F(Y, \eta)|^2 dY d\eta \right). \end{aligned} \quad (7.45)$$

For $|\eta| < 1$, we have

$$\begin{aligned} & \sum_{\nu=0}^{l-|\alpha|} \sum_{|\beta|=\nu} \frac{|\eta^\beta|^2}{|(\eta, p)|^{2|\beta|}} (|Y|^{2(|\alpha|+|\beta|-l)} + |(\eta, p)|^{2(|\alpha|+|\beta|-l)}) \\ & \geq |Y|^{2(|\alpha|-l)} + |(\eta, p)|^{2(|\alpha|-l)} \geq k_5 (|Y|^{2(|\alpha|-l)} + 1), \end{aligned}$$

where $k_5 > 0$ does not depend on Y , η , and p . Therefore,

$$\begin{aligned} & \sum_{|\alpha| \leq l} \sum_{\nu=0}^{l-|\alpha|} \int_{|\eta| < 1} \int_{\Theta_j} \Phi_{\alpha\nu}(\eta, p, Y) |D_Y^\alpha F(Y, \eta)|^2 dY d\eta \\ & \geq k_5 \sum_{|\alpha| \leq l} \int_{|\eta| < 1} \int_{\Theta_j} |(\eta, p)|^{-2(a-l+1)} |Y|^{2a} (|Y|^{2(|\alpha|-l)} + 1) |D_Y^\alpha F(Y, \eta)|^2 dY d\eta. \end{aligned} \quad (7.46)$$

For $|\eta| > 1$, we have

$$\sum_{|\beta|=\nu} \frac{|\eta^\beta|^2}{|(\eta, p)|^{2|\beta|}} \geq k_6 \quad (p \in S^{d-1}, \nu = 0, \dots, l), \quad (7.47)$$

where $k_6 > 0$ does not depend on η and p . It follows from (7.43) and (7.47) that

$$\begin{aligned} & \sum_{|\alpha| \leq l} \sum_{\nu=0}^{l-|\alpha|} \int_{|\eta| > 1} \int_{\Theta_j} \Phi_{\alpha\nu}(\eta, p, Y) |D_Y^\alpha F(Y, \eta)|^2 dY d\eta \\ & \geq k_6 \sum_{|\alpha| \leq l} \sum_{\nu=0}^{l-|\alpha|} \int_{|\eta| > 1} \int_{\Theta_j} |(\eta, p)|^{-2(a-l+1)} |Y|^{2(a+|\alpha|+\nu-l)} |D_Y^\alpha F(Y, \eta)|^2 dY d\eta \\ & \geq k_6 \sum_{|\alpha| \leq l} \int_{|\eta| > 1} \int_{\Theta_j} |(\eta, p)|^{-2(a-l+1)} |Y|^{2a} (|Y|^{2(|\alpha|-l)} + 1) |D_Y^\alpha F(Y, \eta)|^2 dY d\eta. \end{aligned} \quad (7.48)$$

Inequalities (7.45), (7.46), and (7.48) imply that

$$\|f_j\|_{E_a^l(\Theta_j)}^2 \geq k_7 \int_{\mathbb{R}^{n-2}} |(\eta, p)|^{-2(a-l+1)} \|F_j(\cdot, \eta)\|_{E_a^l(\Theta_j)}^2 d\eta, \quad (7.49)$$

where $k_7 > 0$ does not depend on p .

Relation (7.37) follows from (7.44) and (7.49). Similarly, one can prove (7.38). \square

Now we can prove Theorem 7.1.

Proof of Theorem 7.1. It is easy to see that $u(x)$ is a solution of the problem

$$\begin{aligned} A_j(D_y, D_z, p)u_j(y, z) &= f_j(y, z), \quad (y, z) \in \Theta_j, \\ B_{j\rho\mu}(p)u &\equiv \sum_{k,s} (B_{j\rho\mu ks}(D_y, D_z, p)u_k)(\mathcal{G}_{j\rho ks}y, z)|_{\Gamma_{j\rho}} = f_{j\rho\mu}(y, z), \quad (y, z) \in \Gamma_{j\rho}, \end{aligned}$$

iff the function $v(x) = u(|p|^{-1}x)$ is a solution of the problem

$$\begin{aligned} A_j(D_y, D_z, p|p|^{-1})v_j(y, z) &= |p|^{-2m} f_j(|p|^{-1}y, |p|^{-1}z), \quad (y, z) \in \Theta_j, \\ B_{j\rho\mu}(p|p|^{-1})v &\equiv \sum_{k,s} (B_{j\rho\mu ks}(D_y, D_z, p|p|^{-1})v_k)(\mathcal{G}_{j\rho ks}y, z)|_{\Gamma_{j\rho}} \\ &= |p|^{-m_{j\rho\mu}} f_{j\rho\mu}(|p|^{-1}y, |p|^{-1}z), \quad (y, z) \in \Gamma_{j\rho}, \end{aligned}$$

where $p \in \mathbb{R}^d \setminus \{0\}$.

Further, we shall use the following inequalities:

$$\|v_j\|_{E_a^{l+2m}(\Theta_j)}^2 \geq |p|^{2a+n-2(l+2m)} \|u_j\|_{V_a^{l+2m}(\Theta_j)}^2, \quad p \in \mathbb{R}^d \setminus \{0\}, \quad (7.50)$$

$$\begin{aligned} \|A_j(D_y, D_z, p|p|^{-1})v_j\|_{E_a^l(\Theta_j)}^2 \\ \leq k_1 |p|^{2a+n-2(l+2m)} \|A_j(D_y, D_z, p)u_j\|_{V_a^l(\Theta_j)}^2, \quad p \in \mathbb{R}^d \setminus \{0\}, \end{aligned} \quad (7.51)$$

$$\begin{aligned} \|B_{j\rho\mu}(p|p|^{-1})v\|_{E_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})}^2 \\ \leq k_2 |p|^{2a+n-2(l+2m)} \|B_{j\rho\mu}(p)u\|_{V_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})}^2, \quad p \in \mathbb{R}^d \setminus \{0\}, \end{aligned} \quad (7.52)$$

where $k_1, k_2 > 0$ do not depend on u and p . To obtain inequality (7.50), we introduce the new variables $x' = |p|^{-1}x$ and $y' = |p|^{-1}y$, where $x = (y, z)$, $x' = (y', z')$, $y, y' \in \mathbb{R}^2$, and $z, z' \in \mathbb{R}^{n-2}$. Then we have

$$\begin{aligned} \|v_j\|_{E_a^{l+2m}(\Theta_j)}^2 &= \sum_{|\alpha| \leq l+2m} \int_{\Theta_j} |y|^{2a} (|y|^{2(|\alpha|-l-2m)} + 1) |D_x^\alpha u_j(|p|^{-1}x)|^2 dx \\ &= |p|^{2a+n-2(l+2m)} \sum_{|\alpha| \leq l+2m} \int_{\Theta_j} |y'|^{2a} (|y'|^{2(|\alpha|-l-2m)} + |p|^{2(l+2m-|\alpha|)}) |D_{x'}^\alpha u_j(x')|^2 dx' \\ &\geq |p|^{2a+n-2(l+2m)} \|u_j\|_{V_a^{l+2m}(\Theta_j)}^2. \end{aligned}$$

Using Lemma 7.1, similarly to (7.50), we derive (7.51).

To obtain inequality (7.52), one can use the equivalent norms in $E_a^{k-1/2}(\Gamma_{j\rho})$ and $H_a^{k-1/2}(\Gamma_{j\rho})$ given by

$$\begin{aligned} \|u\|'_{E_a^{k-1/2}(\Gamma_{j\rho})} &= \left(\sum_{|\alpha|=k-1} \int_{\Gamma_{j\rho}} \int_{\Gamma_{j\rho}} |y_1|^a D^\alpha u(x_1) - |y_2|^a D^\alpha u(x_2)|^2 \frac{dx_1 dx_2}{|x_1 - x_2|^n} \right. \\ &\quad \left. + \sum_{|\alpha| \leq k-1} \int_{\Gamma_{j\rho}} |y|^{2a} (|y|^{2(|\alpha|-k+1/2)} + 1) |D^\alpha u(x)|^2 dx \right)^{1/2}, \end{aligned}$$

$$\|u\|'_{H_a^{k-1/2}(\Gamma_{j\rho})} = \left(\sum_{|\alpha|=k-1} \int_{\Gamma_{j\rho}} \int_{\Gamma_{j\rho}} | |y_1|^a D^\alpha u(x_1) - |y_2|^a D^\alpha u(x_2) |^2 \frac{dx_1 dx_2}{|x_1 - x_2|^n} + \sum_{|\alpha| \leq k-1} \int_{\Gamma_{j\rho}} |y|^{2(a+|\alpha|-k+1/2)} |D^\alpha u(x)|^2 dx \right)^{1/2}$$

(see Lemmas 9.1 and 1.3 in [20]) and Lemma 7.3.

Now the assertion of the theorem follows from Lemma 7.5 and inequalities (7.50), (7.51), and (7.52). \square

Further, we prove an analog of Corollary 2.1. Introduce the linear bounded operator $\mathcal{L}_g''(p) : \mathcal{V}_a^{l+2m}(\Theta) \rightarrow \mathcal{V}_a^l(\Theta, \Gamma)$ by the formula

$$\mathcal{L}_g''(p)v = \mathcal{L}_g(p)v + \eta(\mathcal{L}_g'(p) - \mathcal{L}_g(p))v,$$

where η is the same function as in Sec. 2.3.

Corollary 7.1. *Let the conditions of Theorem 7.1 hold. Then the operator $\mathcal{L}_g''(p) : \mathcal{V}_a^{l+2m}(\Theta) \rightarrow \mathcal{V}_a^l(\Theta, \Gamma)$ is an isomorphism for all sufficiently small $\delta > 0$ and $p \in \mathbb{R}^d \setminus \{0\}$ and*

$$\|u\|_{\mathcal{V}_a^{l+2m}(\Theta)} \leq c \|\mathcal{L}_g''(p)u\|_{\mathcal{V}_a^l(\Theta, \Gamma)}, \quad p \in \mathbb{R}^d \setminus \{0\},$$

where $c > 0$ does not depend on u and p .

Proof. By Theorem 7.1, there is a bounded operator $\mathcal{L}_g^{-1}(p)$ and estimate (7.30) holds. We have

$$\mathcal{L}_g''(p)\mathcal{L}_g^{-1}(p) = \mathcal{I} + \mathcal{M}(p),$$

where \mathcal{I} denotes the identity operator on $\mathcal{V}_a^l(\Theta, \Gamma)$ and

$$\mathcal{M}(p) = \eta(\mathcal{L}_g'(p) - \mathcal{L}_g(p))\mathcal{L}_g^{-1}(p).$$

It follows from (7.14) that

$$\|u\|_{\mathcal{G}_{j\rho ks}(\Gamma_i)} \|V_a^{l+2m-m_{j\rho\mu}-1/2}(\mathcal{G}_{j\rho ks}(\Gamma_i))\| \leq k_1 \|u\|_{V_a^{l+2m-m_{j\rho\mu}}(\Theta_k)},$$

where $k_1, k_2, \dots > 0$ do not depend on u and p . Therefore, similarly to the proof of Corollary 2.1, we obtain

$$\|\mathcal{M}(p)f\|_{\mathcal{V}_a^l(\Theta, \Gamma)} \leq k_2 \delta \|\mathcal{L}_g^{-1}(p)f\|_{\mathcal{V}_a^{l+2m}(\Theta)}.$$

Combining this inequality with (7.30) yields

$$\|\mathcal{M}(p)f\|_{\mathcal{V}_a^l(\Theta, \Gamma)} \leq k_3 \delta \|f\|_{\mathcal{V}_a^l(\Theta, \Gamma)}.$$

If $\delta > 0$ is so small that $k_3 \delta \leq 1/2$, then there exists the inverse operator $(\mathcal{I} + \mathcal{M}(p))^{-1}$ bounded in the norms $\|\cdot\|_{\mathcal{V}_a^l(\Theta, \Gamma)}$ uniformly with respect to $p \in \mathbb{R}^d \setminus \{0\}$.

Clearly, the operator $\mathcal{L}_g^{-1}(p)(\mathcal{I} + \mathcal{M}(p))^{-1}$ is the right inverse for $\mathcal{L}_g''(p)$ and

$$\|\mathcal{L}_g^{-1}(p)(\mathcal{I} + \mathcal{M}(p))^{-1}f\|_{\mathcal{V}_a^{l+2m}(\Theta)} \leq k_4 \|f\|_{\mathcal{V}_a^l(\Theta, \Gamma)}, \quad p \in \mathbb{R}^d \setminus \{0\}.$$

Similarly, one can prove the existence of a left inverse operator for $\mathcal{L}_g''(p)$. \square

7.4

Now we fix an arbitrary point $g \in K_2$. Similarly to Sec. 1.4, we arrive at the following model operator:

$$\begin{aligned} \mathcal{L}_g(p) : V_a^{l+2m}(\mathbb{R}_+^n) &\rightarrow \mathcal{V}_a^l(\mathbb{R}_+^n, \Gamma) \\ &= V_a^l(\mathbb{R}_+^n) \times V_a^{l+2m-m_{i\mu}-1/2}(\mathbb{R}_-^{n-1}) \times V_a^{l+2m-m_{i\mu}-1/2}(\mathbb{R}_+^{n-1}) \end{aligned}$$

given by

$$\mathcal{L}_g(p)u = (A(D_y, D_z, p)u, B_{i\mu 0}(D_y, D_z, p)u|_{\varphi=-\pi/2}, B_{i\mu 0}(D_y, D_z, p)u|_{\varphi=\pi/2})$$

(cf. (1.16)). We assume that the space $V_a^{l+2m}(\mathbb{R}_+^n)$ is equipped with the norm (7.3) and $\mathcal{V}_a^l(\mathbb{R}_+^n, \Gamma)$ with the norm

$$\|f\|_{\mathcal{V}_a^l(\mathbb{R}_+^n, \Gamma)} = \left(\|f_0\|_{V_a^l(\mathbb{R}_+^n)}^2 + \|f_-\|_{V_a^{l+2m-m_{i\mu}-1/2}(\mathbb{R}_-^{n-1})}^2 + \|f_+\|_{V_a^{l+2m-m_{i\mu}-1/2}(\mathbb{R}_+^{n-1})}^2 \right)^{1/2},$$

where $f = (f_0, f_-, f_+)$ and the norm $\|f_\pm\|_{V_a^{l+2m-m_{i\mu}-1/2}(\mathbb{R}_\pm^{n-1})}$ is defined by (7.4).

Analogously to Sec. 2.5, we consider the linear bounded operator

$$\mathcal{L}_g(\omega) : E_a^{l+2m}(\mathbb{R}_+^2) \rightarrow \mathcal{E}_a^l(\mathbb{R}_+^2, \gamma)$$

given by

$$\mathcal{L}_g(\omega)V = (A(D_y, \omega)V, B_{i\mu 0}(D_y, \omega)V|_{\mathbb{R}_-}, B_{i\mu 0}(D_y, \omega)V|_{\mathbb{R}_+}),$$

where $\omega = (\eta, p)/|(\eta, p)| \in S^{n+d-3}$ (cf. (2.26) and (7.29)).

Finally, we consider the analytic operator-valued function

$$\hat{\mathcal{L}}_g(\lambda) : W^{l+2m}(-\pi/2, \pi/2) \rightarrow \mathcal{W}^l[-\pi/2, \pi/2]$$

given by (2.27).

The following theorem is an analog of Theorem 2.3 (cf. Theorem 9.1 and Corollary 9.1 in [20]).

Theorem 7.2. *Let Conditions 7.1 and 7.2 hold. Assume that the line $\text{Im } \lambda = a + 1 - l - 2m$ contains no eigenvalues of $\hat{\mathcal{L}}_g(\lambda)$ and $\dim \mathcal{N}(\mathcal{L}_g(\omega)) = \text{codim } \mathcal{R}(\mathcal{L}_g(\omega)) = 0$ for any $\omega \in S^{n+d-3}$. Then the operator $\mathcal{L}_g(p) : \mathcal{V}_a^{l+2m}(\mathbb{R}_+^n) \rightarrow \mathcal{V}_a^l(\mathbb{R}_+^n, \Gamma)$ is an isomorphism for $p \in \mathbb{R}^d \setminus \{0\}$ and*

$$\|u\|_{\mathcal{V}_a^{l+2m}(\mathbb{R}_+^n)} \leq c \|\mathcal{L}_g(p)u\|_{\mathcal{V}_a^l(\mathbb{R}_+^n, \Gamma)}, \quad p \in \mathbb{R}^d \setminus \{0\},$$

where $c > 0$ does not depend on u and p .

The proof is similar to the proof of Theorem 7.1.

8 Solvability of Nonlocal Elliptic Problems with a Parameter

8.1

In this section, we prove the unique solvability of nonlocal elliptic problems with a parameter in bounded domains.

Lemma 8.1. *Let H be a Hilbert space and I the identity operator on H . Let $M_\varepsilon(p)$ and $S_\varepsilon(p)$ ($\varepsilon > 0$, $p \in \mathbb{R}^d$, and $|p|$ is sufficiently large) be families of bounded operators on H , such that*

$$\|M_\varepsilon(p)\| \leq c_1\varepsilon, \quad \|S_\varepsilon(p)\| \leq c_2, \quad \|S_\varepsilon^2(p)\| \leq c_3|p|^{-1}, \quad (8.1)$$

where $c_1, c_2, c_3 > 0$ do not depend on ε and p . Then the operators

$$L_\varepsilon(p) = I + M_\varepsilon(p) + S_\varepsilon(p)$$

have bounded inverse operators $L_\varepsilon^{-1}(p)$ and the estimate

$$\|L_\varepsilon^{-1}(p)\| \leq c_4$$

holds for sufficiently small $\varepsilon > 0$ and sufficiently large $|p|$, where $c_4 > 0$ does not depend on ε and p .

Proof. To prove the lemma, we will construct a right and a left inverse operators for $L_\varepsilon(p)$. We have

$$L_\varepsilon(p)(I - (M_\varepsilon(p) + S_\varepsilon(p))) = I - M_\varepsilon^2(p) - M_\varepsilon(p)S_\varepsilon(p) - S_\varepsilon(p)M_\varepsilon(p) - S_\varepsilon^2(p).$$

It follows from (8.1) that

$$\|S_\varepsilon^2(p)\| \leq 1/6$$

for sufficiently large $|p|$ and

$$\|M_\varepsilon(p)\| \leq \min(1/\sqrt{6}, 1/(12c_2))$$

for sufficiently small ε . Therefore,

$$\|M_\varepsilon^2(p) + M_\varepsilon(p)S_\varepsilon(p) + S_\varepsilon(p)M_\varepsilon(p) + S_\varepsilon^2(p)\| \leq 1/2.$$

Thus, the operators $(I - M_\varepsilon^2(p) - M_\varepsilon(p)S_\varepsilon(p) - S_\varepsilon(p)M_\varepsilon(p) - S_\varepsilon^2(p))^{-1}$ exist and are uniformly bounded with respect to ε and p . Combining this fact with the uniform boundedness of the operators $I - (M_\varepsilon(p) + S_\varepsilon(p))$, we see that the operators

$$L_\varepsilon^{-1}(p) = (I - (M_\varepsilon(p) + S_\varepsilon(p)))(I - M_\varepsilon^2(p) - M_\varepsilon(p)S_\varepsilon(p) - S_\varepsilon(p)M_\varepsilon(p) - S_\varepsilon^2(p))^{-1}$$

are the right inverse for the operators $L_\varepsilon(p)$ and

$$\|L_\varepsilon^{-1}(p)\| \leq c_4.$$

Similarly, one can prove that there exist uniformly bounded left inverse operators for the operators $L_\varepsilon(p)$. \square

Lemma 8.2. *Let Conditions 7.1, 7.2, 1.3, and 1.4 hold. Assume that the line $\text{Im } \lambda = a + 1 - l - 2m$ contains no eigenvalues of $\hat{\mathcal{L}}_g(\lambda)$ for any $g \in K$ and $\dim \mathcal{N}(\mathcal{L}_g(\omega)) = \text{codim } \mathcal{R}(\mathcal{L}_g(\omega)) = 0$ for any $g \in K$ and $\omega \in S^{n+d-3}$. Then there is a number $p_0 > 0$ such that the operator $\mathbf{L}^1(p)$, $|p| \geq p_0$, has a bounded inverse and*

$$c_1 \|\mathbf{L}^1(p)u\|_{\mathcal{H}_a^l(Q, \Gamma)} \leq \|u\|_{H_a^{l+2m}(Q)} \leq c_2 \|\mathbf{L}^1(p)u\|_{\mathcal{H}_a^l(Q, \Gamma)}, \quad |p| \geq p_0, \quad (8.2)$$

where $c_1, c_2 > 0$ do not depend on u and p .

Proof. 1. The first inequality in (8.2) follows from the definition of the norms $\|\cdot\|$, from Lemma 7.1, and from estimate (7.15). To prove the second inequality in (8.2), we repeat the proof of Lemma 4.3, replacing there the norms $\|\cdot\|$ by the norms $\|\cdot\|$, Corollary 2.1 and Theorem 2.3 by Corollary 7.1 and Theorem 7.2, respectively, and the results on elliptic problems in the interior of the domain and near a smooth part of the boundary by the corresponding results on elliptic problems with a parameter [2] and taking into account estimate (7.15). Then we obtain the following a priori estimate:

$$\|u\|_{H_a^{l+2m}(Q)} \leq k_1 (\|\mathbf{L}^1(p)u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \|u\|_{H_a^{l+2m-1}(Q)}), \quad p \in \mathbb{R}^d \setminus \{0\},$$

where $k_1 > 0$ does not depend on u and p . Combining this estimate with the relation

$$\|u\|_{H_a^{l+2m-1}(Q)} \leq |p|^{-1} \|u\|_{H_a^{l+2m}(Q)}$$

and taking $|p| \geq p'$, where $p' > 0$ is sufficiently large, we obtain the second inequality in (8.2).

2. It remains to prove the existence of a right inverse operator for $\mathbf{L}^1(p)$.

Using the notation from the proof of Lemma 5.2, we introduce the operator

$$R_{\mathcal{K}_1}(p)f = \sum_t (U^t)^{-1} \left(\hat{\xi}^t(\mathcal{L}_{g^t}''(p))^{-1} F^t \left(\sum_q \xi_q^t f \right) \right)$$

(cf. (5.9)). Similarly to (5.24), we prove that

$$\mathbf{L}^1(p)R_{\mathcal{K}_1}(p)f = \xi_0 f + T_{\mathcal{K}_1}(p)f, \quad (8.3)$$

where $T_{\mathcal{K}_1}(p) : \mathcal{H}_a^l(Q, \Gamma) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ is a bounded operator such that

$$\|T_{\mathcal{K}_1}(p)f\|_{\mathcal{H}_a^l(Q, \Gamma)} \leq c_1 \|f\|_{\mathcal{H}_a^l(Q, \Gamma)}, \quad (8.4)$$

$$\|T_{\mathcal{K}_1}^2(p)f\|_{\mathcal{H}_a^l(Q, \Gamma)} \leq c_2 |p|^{-1} \|f\|_{\mathcal{H}_a^l(Q, \Gamma)}, \quad (8.5)$$

and $c_1, c_2, \dots > 0$ do not depend on f , p , and on the number ε in the definition of the function ξ , $|p| \geq p'$.

Estimate (8.4) follows from Corollary 7.1 and inequalities (7.6) and (7.15). Let us prove (8.5). By analogy with the operators $\mathcal{T}_{j\rho\mu}^t$, A_k'' , and T_i , $i = 1, 2, 3$, from the proof of Lemma 5.2, we consider the corresponding operators $\mathcal{T}_{j\rho\mu}^t(p)$, $A_k''(p)$ and $T_i(p)$, $i = 1, 2, 3$, depending on the parameter p . First, we estimate the norm of $T_3^2(p)f$. Introduce functions $\psi_k^t \in C_0^\infty(\Theta_k)$ such that $\psi_k^t(x') = 1$ for $x' \in \Omega_k^t$. Using inequality (5.20) with the norms $\|\cdot\|$ replaced by the norms $\|\cdot\|$, the equivalence of the norms $\|\cdot\|$ in the subspaces of $H_a^l(\Theta_k)$ and $W^l(\Theta_k)$ consisting of compactly supported functions vanishing near the edge \mathcal{P} , Theorem 4.1 in [2], equality (5.22), Leibniz' formula, inequality (7.6), and Corollary 7.1, we have

$$\begin{aligned} & \left\| \mathcal{T}_{j\rho\mu}^t(p)(\mathcal{L}_{g^t}''(p))^{-1} F^t \left(\sum_q \xi_q^t T_3(p)f \right) \right\|_{V_a^{l+2m-m_{j\rho\mu}-1/2}(\Gamma_{j\rho})} \\ & \leq k_1 \sum_k \left\| A_k''(p) \left(\psi_k^t \left[(\mathcal{L}_{g^t}''(p))^{-1} F^t \left(\sum_q \xi_q^t T_3(p)f \right) \right]_k \right) \right\|_{V_a^l(\Theta_k)} \\ & \leq k_2 \left\| (\mathcal{L}_{g^t}''(p))^{-1} F^t \left(\sum_q \xi_q^t T_3(p)f \right) \right\|_{V_a^{l+2m-1}(\Theta)} \leq k_3 |p|^{-1} \|f\|_{\mathcal{H}_a^l(Q, \Gamma)}, \end{aligned}$$

where $p \in \mathbb{R}^d \setminus \{0\}$ and $k_1, \dots, k_4 > 0$ do not depend on f , p , and ε .

The latter inequality implies that

$$\|T_3^2(p)f\|_{\mathcal{H}_a^l(Q,\Gamma)} \leq k_4|p|^{-1}\|f\|_{\mathcal{H}_a^l(Q,\Gamma)}, \quad p \in \mathbb{R}^d \setminus \{0\}.$$

Similarly, we estimate the norm of $T_1^2(p)f$. The analogous estimate for $T_2(p)f$ is evident. Thus, we obtain inequality (8.5) for $T_{\mathcal{K}_1}(p) = T_1(p) + T_2(p) + T_3(p)$.

Let ζ be a function defined in (5.28). Set $\zeta_1 = 1 - \zeta$. Since $\zeta(x) = 1$ for $x \in \mathcal{K}_1^{2\varepsilon}$, it follows that $\text{supp } \zeta_1 \subset \overline{Q} \setminus \mathcal{K}_1^{2\varepsilon}$. Introduce a function $\hat{\zeta}_1 \in C^\infty(\mathbb{R}^n)$ such that $\hat{\zeta}_1(x) = 1$ for $x \in \overline{Q} \setminus \mathcal{K}_1^{2\varepsilon}$ and $\text{supp } \hat{\zeta}_1 \subset \overline{Q} \setminus \mathcal{K}_1^\varepsilon$.

Due to Theorem 10.4 in [20], there exists a bounded operator $\mathbf{R}^0(p)$ such that

$$\mathbf{L}_0(p)\mathbf{R}_0(p)f = f$$

for $f \in \mathcal{H}_a^l(Q, \Gamma)$, $\text{supp } f \subset \overline{Q} \setminus \mathcal{K}_1^{2\varepsilon}$, provided that $|p| \geq p''$, where $p'' \geq p'$ is sufficiently large. Thus, we can set

$$R(p)f = R_{\mathcal{K}_1}(p)f + R_{\mathcal{K}_1}(p)(\eta f) + \hat{\zeta}_1 \mathbf{R}^0(p)(\zeta_1 f),$$

where $\eta(x) = \zeta_0(x)(1 - \xi_0(x))/\xi_0(x)$ for $x \in \mathcal{K}_1^{4\varepsilon}$ and $\eta(x) = 0$ for $x \notin \mathcal{K}_1^{4\varepsilon}$ (cf. (5.30)). Since $\text{supp } \hat{\zeta}_1 \subset \overline{Q} \setminus \mathcal{K}_1^\varepsilon$, we have

$$B_{i\mu}^1(p)(\hat{\zeta}_1 \mathbf{R}^0(p)(\zeta_1 f)) = 0$$

and hence

$$\mathbf{L}^1(p)R(p)f = \mathbf{L}^1(p)R_{\mathcal{K}_1}(p)f + \mathbf{L}^1(p)R_{\mathcal{K}_1}(p)(\eta f) + \mathbf{L}^0(p)(\hat{\zeta}_1 \mathbf{R}^0(p)(\zeta_1 f)). \quad (8.6)$$

Combining this relation with (8.3) and using Leibniz' formula and Lemmas 7.1 and 7.2, we obtain

$$\begin{aligned} \mathbf{L}^1(p)R(p)f &= \xi_0 f + T_{\mathcal{K}_1}(p)f + \zeta_0(1 - \xi_0)f + T_{\mathcal{K}_1}(p)(\eta f) + \zeta_1 f + T(p)f \\ &= f + T_{\mathcal{K}_1}(p)f + M(p)f + T(p)f \end{aligned}$$

or, equivalently,

$$\mathbf{L}^1(p)R(p) = \mathbf{I} + T_{\mathcal{K}_1}(p) + M(p) + T(p), \quad (8.7)$$

where

$$M(p)f = T_{\mathcal{K}_1}(p)(\eta f),$$

while $T(p) : \mathcal{H}_a^l(Q, \Gamma) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ is a bounded operator such that

$$\|T(p)f\|_{\mathcal{H}_a^l(Q,\Gamma)} \leq k_1|p|^{-1}\|f\|_{\mathcal{H}_a^l(Q,\Gamma)}, \quad (8.8)$$

where $k_1 = k_1(\varepsilon) > 0$ does not depend on f and p .

By inequality (8.4), we have

$$\|M(p)f\|_{\mathcal{H}_a^l(Q,\Gamma)} \leq c_3\|\eta f\|_{\mathcal{H}_a^l(Q,\Gamma)}.$$

However, $(1 - \xi_0(x))/\xi_0(x) = 0$ for $x \in \mathcal{K}_1$, while the function ζ_0 is supported in $\mathcal{K}_1^{4\varepsilon}$ and satisfies the inequality in (5.27). Therefore, it follows from the last estimate, from Lemmas 4.1 and 4.2 and from Remark 4.1 that

$$\|M(p)f\|_{\mathcal{H}_a^l(Q,\Gamma)} \leq c_4\varepsilon\|f\|_{\mathcal{H}_a^l(Q,\Gamma)}. \quad (8.9)$$

By virtue of inequalities (8.4), (8.5), and (8.9) and Lemma 8.1, the operator

$$(\mathbf{I} + T_{\mathcal{K}_1}(p) + M(p))^{-1} \quad (8.10)$$

exists and is bounded in the norms $\|\cdot\|_{\mathcal{H}_a^l(Q,\Gamma)}$, uniformly with respect to p , $|p| \geq p'''$, where $p''' \geq p''$ is sufficiently large, provided that $\varepsilon > 0$ is a sufficiently small fixed number. Therefore, relation (8.7) is equivalent to the following one:

$$\mathbf{L}^1(p)R(p)(\mathbf{I} + T_{\mathcal{K}_1}(p) + M(p))^{-1} = \mathbf{I} + T'(p),$$

where

$$T'(p) = T(p)(\mathbf{I} + T_{\mathcal{K}_1}(p) + M(p))^{-1}.$$

By virtue of the uniform boundedness of the operator (8.10) and estimate (8.8), there is a sufficiently large number $p_0 \geq p'''$ such that $\|T'(p)\| \leq 1/2$ for $|p| \geq p_0$ (recall that ε is fixed) and hence

$$\mathbf{L}^1(p)R(p)(\mathbf{I} + T_{\mathcal{K}_1}(p) + M(p))^{-1}(\mathbf{I} + T'(p))^{-1} = \mathbf{I}.$$

Thus, we have proved the existence of the right inverse operator for $\mathbf{L}^1(p)$, $|p| \geq p_0$. Combining this with the second estimate in (8.2), we complete the proof. \square

8.2

In this subsection, we generalize the result of the previous subsection to the operator $\mathbf{L}(p)$.

Theorem 8.1. *Let Conditions 7.1–7.4, 1.3, and 1.4 hold. Assume that the line $\text{Im } \lambda = a + 1 - l - 2m$ contains no eigenvalues of $\hat{\mathcal{L}}_g(\lambda)$ for any $g \in K$ and $\dim \mathcal{N}(\mathcal{L}_g(\omega)) = \text{codim } \mathcal{R}(\mathcal{L}_g(\omega)) = 0$ for any $g \in K$ and $\omega \in S^{n+d-3}$. Then the following assertions are true:*

1. *there is a number $p_1 > 0$ such that the operator $\mathbf{L}(p)$, $|p| \geq p_1$, has a bounded inverse and*

$$c_1 \|\mathbf{L}(p)u\|_{\mathcal{H}_a^l(Q, \Gamma)} \leq \|u\|_{H_a^{l+2m}(Q)} \leq c_2 \|\mathbf{L}(p)u\|_{\mathcal{H}_a^l(Q, \Gamma)}, \quad |p| \geq p_1, \quad (8.11)$$

where $c_1, c_2 > 0$ do not depend on u and p ;

2. *the operator $\mathbf{L}(p)$ has the Fredholm property and $\text{ind } \mathbf{L}(p) = 0$ for $p \in \mathbb{R}^d$.*

To prove Theorem 8.1, we preliminarily consider the operators

$$L_t(p) = \mathbf{L}^1(p) + t(\mathbf{L}(p) - \mathbf{L}^1(p)), \quad 0 \leq t \leq 1.$$

Clearly, $L_0(p) = \mathbf{L}^1(p)$, $L_1(p) = \mathbf{L}(p)$.

Lemma 8.3. *Let the conditions of Theorem 8.1 hold. Then there is a number $p_1 > 0$ such that the following estimates hold for $u \in H_a^{l+2m}(Q)$:*

$$c_1 \|L_t(p)u\|_{\mathcal{H}_a^l(Q, \Gamma)} \leq \|u\|_{H_a^{l+2m}(Q)} \leq c_2 \|L_t(p)u\|_{\mathcal{H}_a^l(Q, \Gamma)}, \quad |p| \geq p_1, \quad 0 \leq t \leq 1, \quad (8.12)$$

where $c_1, c_2 > 0$ do not depend on u , p , and t .

Proof. The definition of the norms $\|\cdot\|$, Lemma 7.1, inequality (7.15), and Conditions 7.3 and 7.4 imply the first estimate in (8.12).

Let us prove the second estimate in (8.12). Applying Lemma 8.2 and using Condition 7.3 and the fact that $0 \leq t \leq 1$, we have

$$\begin{aligned} \|u\|_{H_a^{l+2m}(Q)} &\leq k_1 \|L_0(p)u\|_{\mathcal{H}_a^l(Q, \Gamma)} \\ &\leq k_2 (\|L_t(p)u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \sum_{i, \mu} \|B_{i\mu}^2(p)\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} + \|u\|_{H_a^{l+2m-1}(Q)}), \end{aligned} \quad (8.13)$$

where $|p| \geq p_0$ and $k_1, k_2, \dots > 0$ do not depend on u , p , and t .

Further, we can repeat the proof of inequalities (4.35) and (4.36), using Lemma 7.1 and estimate (7.15) and replacing the operators $B_{i\mu}^2$ by $B_{i\mu}^2(p)$, the norms $\|\cdot\|$ by the norms $\|\cdot\|$, Lemma 4.6 by Condition 7.4, and the results on elliptic problems in the interior of the domain and near a smooth part of boundary by the corresponding results for elliptic problems with a parameter [2]. Thus, we obtain

$$\|B_{i\mu}^2(p)u\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Gamma_i)} \leq k_3 (\|L_t(p)u\|_{\mathcal{H}_a^l(Q, \Gamma)} + \|u\|_{H_a^{l+2m-1}(Q)}). \quad (8.14)$$

Combining estimates (8.13) and (8.14) with Lemma 7.1 and taking $|p| \geq p_1$, where $p_1 \geq p_0$ is sufficiently large, we complete the proof. \square

Now we will prove Theorem 8.1, using Lemmas 8.2 and 8.3, and the method of continuation along the parameter t .

Proof of Theorem 8.1. 1. Applying Lemmas 8.2 and 8.3, we see that the operator

$$L_t(p) = L_0(p)(\mathbf{I} + tL_0^{-1}(p)(L_1(p) - L_0(p)))$$

has a bounded inverse for $0 \leq t \leq t_1 = c_1/(4c_2)$ with the norm $\|L_t^{-1}(p)\| \leq c_2$. Therefore, the operator

$$L_t(p) = L_{t_1}(p)(\mathbf{I} + (t - t_1)L_{t_1}^{-1}(p)(L_1(p) - L_0(p)))$$

has a bounded inverse for $t_1 \leq t \leq 2t_1$ with the norm $\|L_t^{-1}(p)\| \leq c_2$. Repeating this procedure finitely many times, we see that the operator $L_1(p) = \mathbf{L}(p)$ has a bounded inverse. Estimate (8.11) follows from (8.12) for $t = 1$.

2. Fix $\hat{p} \in \mathbb{R}^d$ such that $|\hat{p}| \geq p_0$. In this case, there exists a bounded operator $(\mathbf{L}^1(\hat{p}))^{-1} : \mathcal{H}_a^l(Q, \Gamma) \rightarrow H_a^{l+2m}(Q)$ due to Lemma 8.2. Thus, we have

$$\mathbf{L}^1(p) = \mathbf{L}^1(\hat{p})(\mathbf{I} + \mathbf{T}(p)),$$

where

$$\mathbf{T}(p) = (\mathbf{L}^1(\hat{p}))^{-1}(\mathbf{L}^1(p) - \mathbf{L}^1(\hat{p})).$$

Clearly, the operator $\mathbf{L}^1(p) - \mathbf{L}^1(\hat{p}) : H_a^{l+2m-1}(Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ is bounded. It follows from the compactness of the embedding $H_a^{l+2m}(Q) \subset H_a^{l+2m-1}(Q)$ that the operator $\mathbf{L}^1(p) - \mathbf{L}^1(\hat{p}) : H_a^{l+2m}(Q) \rightarrow \mathcal{H}_a^l(Q, \Gamma)$ is compact. Therefore, the operator $\mathbf{T}(p) : H_a^{l+2m}(Q) \rightarrow H_a^{l+2m}(Q)$ is also compact. By Theorem 13.2 in [18], the operator $\mathbf{I} + \mathbf{T}(p)$ has the Fredholm property and $\text{ind}(\mathbf{I} + \mathbf{T}(p)) = 0$. Now, applying Theorem 12.2 in [18], we see that the operator $\mathbf{L}^1(p)$ has the Fredholm property and

$$\text{ind} \mathbf{L}^1(p) = \text{ind} \mathbf{L}^1(\hat{p}) + \text{ind}(\mathbf{I} + \mathbf{T}(p)) = 0.$$

Finally, we note that the fulfillment of Conditions 7.3 and 7.4 implies the fulfillment of Conditions 6.1 and 6.2, respectively. Therefore, by Theorem 6.3, the operator $\mathbf{L}(p)$ has the Fredholm property and

$$\text{ind} \mathbf{L}(p) = \text{ind} \mathbf{L}^1(p) = 0.$$

□

8.3

In this subsection, we consider an example of an elliptic problem with a parameter, having distributed nonlocal terms and satisfying Conditions 7.1–7.4, 1.3, and 1.4.

Example 8.1. 1. In the notation of Example 6.1, we consider the following nonlocal problem:

$$-\Delta u + e^{ih} p^2 u = f_0(x), \quad x \in Q, \quad (8.15)$$

$$u|_{\Gamma_l} + B_l^1 u + B_l^2 u = f_l(x), \quad x \in \Gamma_l, \quad l = 1, 2, \quad (8.16)$$

where $-\pi/2 < h < \pi/2$ and $p \geq 0$.

For each point $g \in \mathcal{K}_1$, the operator $\mathcal{L}_g(p) = \mathcal{L}(p) : H_a^2(\Theta) \rightarrow \mathcal{H}_a^0(\Theta, \Gamma)$ given by (7.28) takes the form (cf. (6.14))

$$\begin{aligned} \mathcal{L}(p)v = & (-\Delta v + e^{ih} p^2, \\ & v(\varphi, r, z)|_{\Gamma_{11}} - \alpha_1 v(\varphi + \pi/4, r, z)|_{\Gamma_{11}}, \quad v(\varphi, r, z)|_{\Gamma_{12}} - \alpha_2 v(\varphi - \pi/4, r, z)|_{\Gamma_{12}}). \end{aligned}$$

Hence, the operators

$$\begin{aligned} \mathcal{L}_g(\omega) = \mathcal{L}(\omega) : E_a^2(\theta) &\rightarrow \mathcal{E}_a^0(\theta, \gamma), \\ \hat{\mathcal{L}}_g(\lambda) = \hat{\mathcal{L}}(\lambda) : W^2(-\pi/4, \pi/4) &\rightarrow \mathcal{W}^0[-\pi/4, \pi/4] \end{aligned}$$

given by (7.29) and (2.2) have the form

$$\begin{aligned} \mathcal{L}(\omega)V = & (-\Delta_y V + (\omega_1^2 + e^{ih} \omega_2^2)V, \\ & V(\varphi, r)|_{\gamma_{11}} - \alpha_1 V(\varphi + \pi/4, r)|_{\gamma_{11}}, \quad V(\varphi, r)|_{\gamma_{12}} - \alpha_2 V(\varphi - \pi/4, r)|_{\gamma_{12}}) \end{aligned}$$

and

$$\hat{\mathcal{L}}(\lambda)w = (-w_{\varphi\varphi} + \lambda^2 w, \quad w(-\pi/4) - \alpha_1 w(0), \quad w(\pi/4) - \alpha_2 w(0)),$$

respectively, where $\omega = (\omega_1, \omega_2) \in S^1$.

Let the numbers a, α_1, α_2 satisfy the following relations:

$$0 \leq a \leq 2, \quad 0 < |\alpha_1 + \alpha_2| < 2, \quad \pi/4 < \arctan \sqrt{4(\alpha_1 + \alpha_2)^{-2} - 1}. \quad (8.17)$$

2. We claim that there is a number $h_1 = h_1(\alpha_1, \alpha_2) > 0$ such that the operator $\mathcal{L}(\omega)$, $\omega \in S^1$, is an isomorphism for $|h| \leq h_1$. To prove this fact, one must repeat the reasoning of Example 2.1, where the sesquilinear form (2.14) is replaced by the form

$$b_{\mathcal{R}}[u, v] = \int_{\theta} \left(\sum_{i=1,2} (\mathcal{R}_{\theta} u)_{y_i} \overline{v_{y_i}} + (\omega_1^2 + e^{ih} \omega_2^2) \mathcal{R}_{\theta} u \overline{v} \right) dy$$

with the same domain $D(b_{\mathcal{R}}) = \mathring{W}^1(\theta)$. Let us show that this sesquilinear form remains to be a closed sectorial form.

It follows from the Schwarz inequality and from (2.15) that

$$|b_{\mathcal{R}}[u, v]| \leq k_1 \|u\|_{\mathring{W}^1(\theta)} \|v\|_{\mathring{W}^1(\theta)}, \quad (8.18)$$

where $k_1 > 0$ does not depend on u and v .

Introduce the isomorphism $\mathcal{U} : L_2(\theta) \rightarrow L_2(\theta_1) \times L_2(\theta_1)$ by the formula

$$(\mathcal{U}u)_i(y) = u(\varphi + (i-1)d/2, r), \quad y \in \theta_1, \quad i = 1, 2,$$

and let $R_1 = \begin{pmatrix} 1 & \alpha_1 \\ \alpha_2 & 1 \end{pmatrix}$. Then, using (2.15) and (2.17), we obtain

$$\begin{aligned} \operatorname{Re} b_{\mathcal{R}}[u, u] &= \int_{\theta_1} \left\{ \sum_i \left(\frac{(R_1 + R_1^*)}{2} (\mathcal{U}u_{y_i}, \mathcal{U}u_{y_i}) \right)_{\mathbb{C}^2} \right. \\ &\quad \left. + \omega_1^2 \left(\frac{(R_1 + R_1^*)}{2} \mathcal{U}u, \mathcal{U}u \right)_{\mathbb{C}^2} + \omega_2^2 \left(\frac{(e^{ih}R_1 + (e^{ih}R_1)^*)}{2} \mathcal{U}u, \mathcal{U}u \right)_{\mathbb{C}^2} \right\} dy. \end{aligned} \quad (8.19)$$

Since $|\alpha_1 + \alpha_2| < 2$, it follows that the matrix

$$R_1 + R_1^* = \begin{pmatrix} 2 & \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 & 2 \end{pmatrix} \quad (8.20)$$

is positively definite. Using the Silvester criterion, we will show that the matrix

$$e^{ih}R_1 + (e^{ih}R_1)^* = \begin{pmatrix} e^{ih} + e^{-ih} & e^{ih}\alpha_1 + e^{-ih}\alpha_2 \\ e^{ih}\alpha_2 + e^{-ih}\alpha_1 & e^{ih} + e^{-ih} \end{pmatrix} \quad (8.21)$$

is also positively definite. Since $-\pi/2 < h < \pi/2$, it follows that $e^{ih} + e^{-ih} > 0$. Thus, we have to prove that $\det(e^{ih}R_1 + (e^{ih}R_1)^*) > 0$. Let $e^{ih} = \mu + i\nu$, where $\mu > 0$. Using the equality $\nu^2 = 1 - \mu^2$, we obtain

$$\det(e^{ih}R_1 + (e^{ih}R_1)^*) = 4\mu^2 - [(\alpha_1 + \alpha_2)^2 + (4\mu^2 - 4)\alpha_1\alpha_2].$$

Since $|\alpha_1 + \alpha_2| < 2$, it follows that, for $h = 0$ (i.e., $\mu = 1$),

$$\det(e^{ih}R_1 + (e^{ih}R_1)^*) = 4 - (\alpha_1 + \alpha_2)^2 > 0.$$

Therefore, there is a number $h_1 = h_1(\alpha_1, \alpha_2) > 0$ such that

$$\det(e^{ih}R_1 + (e^{ih}R_1)^*) > 0, \quad |h| \leq h_1.$$

It follows from (8.19), from the positive definiteness of the matrices (8.20) and (8.21), and from the relation $\omega_1^2 + \omega_2^2 = 1$ that

$$\operatorname{Re} b_{\mathcal{R}}[u, u] \geq k_2 \int_{\theta_1} \left\{ \sum_i (\mathcal{U}u_{y_i}, \mathcal{U}u_{y_i})_{\mathbb{C}^2} + (\mathcal{U}u, \mathcal{U}u)_{\mathbb{C}^2} \right\} dy = k_2 \|u\|_{\dot{W}^1(\theta)}^2, \quad (8.22)$$

where $k_2 > 0$ does not depend on u .

Inequalities (8.18) and (8.22) imply that $b_{\mathcal{R}}$ is a closed sectorial form on $L_2(\theta)$, with the domain $D(b_{\mathcal{R}}) = \dot{W}^1(\theta)$ and vertex $k_2 > 0$ (see [13, Chap. 6]).

Thus, repeating the reasoning of Example 2.1, we see that $\mathcal{L}(\omega)$, $\omega \in S^1$, is an isomorphism for the above a, α_1, α_2 , and h . Moreover, since h and ω belong to the compact sets, it follows that the inequality

$$\|V\|_{E_a^2(\theta)} \leq k_3 \|\mathcal{L}(\omega)V\|_{\mathcal{E}_a^0(\theta, \gamma)}, \quad \omega \in S^1, \quad (8.23)$$

holds with a constant $k_3 > 0$ which does not depend⁴ on V , ω , and h .

Now Theorem 8.1 implies that the operator $\mathcal{L}(p)$ is also an isomorphism for $p \geq p_0$, where $p_0 > 0$. Moreover, it follows from (8.23) that the inequality

$$\|u\|_{H_a^2(\Theta)} \leq k_4 \|\mathcal{L}(p)u\|_{\mathcal{H}_a^0(\Theta, \Gamma)}, \quad p > p_0, \quad (8.24)$$

holds with a constant $k_4 > 0$ which does not depend on u and h .

3. Similarly to (6.17) and (6.18), using Lemma 7.1, estimate (7.15), and the interpolation inequalities in Sobolev spaces (see [2, Chap. 1, Sec. 1]), we obtain

$$\|B_l^2 u\|_{H_a^{3/2}(\Gamma_l)} \leq k_5 \|u\|_{H_a^2(Q \setminus \overline{\mathcal{K}_1^{\mathcal{K}}})},$$

$$\|B_l^2 u\|_{H_a^{3/2}(\Gamma_l \setminus \overline{\mathcal{K}_1^{\mathcal{K}}})} \leq k_6 \|u\|_{H_a^2(Q \setminus \mathcal{K})},$$

where $a > 1$ and $k_5, k_6 > 0$ do not depend on u . Thus, the operators B_l^2 satisfy Condition 7.4 for $a > 1$.

⁴Otherwise, denoting $L_h(\omega) = L(\omega)$, we see that there are sequences $h^{(k)}, \omega^{(k)}$, and $V^{(k)}$, $k = 1, 2, \dots$, such that $h^{(k)} \rightarrow h$, $\omega^{(k)} \rightarrow \omega$, $\|\mathcal{L}_{h^{(k)}}(\omega^{(k)})V^{(k)}\|_{\mathcal{E}_a^0(\theta, \gamma)} \rightarrow 0$, and $\|V^{(k)}\|_{E_a^2(\theta)} = 1$. This leads to contradiction because we have $1 = \|V^{(k)}\|_{E_a^2(\theta)} \leq c\|\mathcal{L}_h(\omega)V^{(k)}\|_{\mathcal{E}_a^0(\theta, \gamma)} \leq c(\|\mathcal{L}_{h^{(k)}}(\omega^{(k)})V^{(k)}\|_{\mathcal{E}_a^0(\theta, \gamma)} + \|(\mathcal{L}_{h^{(k)}}(\omega^{(k)})V^{(k)}) - \mathcal{L}_h(\omega)V^{(k)}\|_{\mathcal{E}_a^0(\theta, \gamma)}) \rightarrow 0$.

We consider the linear bounded operators

$$\mathbf{L}(p), \mathbf{L}^1(p) : H_a^2(Q) \rightarrow H_a^0(Q) \times H_a^{3/2}(\Gamma_1) \times H_a^{3/2}(\Gamma_2)$$

given by

$$\mathbf{L}(p)u = \{-\Delta u + e^{ih}p^2u, u|_{\Gamma_l} + B_l^1u + B_l^2u\}, \quad \mathbf{L}^1(p)u = \{-\Delta u + e^{ih}p^2u, u|_{\Gamma_l} + B_l^1u\}.$$

The two results below follow from Lemma 8.2 and Theorem 8.1.

Corollary 8.1. *Let the numbers a, α_1, α_2 satisfy relations (8.17). Then there exist a number $h_1 = h_1(\alpha_1, \alpha_2) > 0$ and a number $p_0 > 0$, independent of h , such that the operator $\mathbf{L}^1(p)$, $p \geq p_0$, $|h| \leq h_1$, has a bounded inverse and*

$$c_1 \|\mathbf{L}^1(p)u\|_{\mathcal{H}_a^0(Q, \Gamma)} \leq \|u\|_{H_a^2(Q)} \leq c_2 \|\mathbf{L}^1(p)u\|_{\mathcal{H}_a^0(Q, \Gamma)},$$

where $c_1, c_2 > 0$ do not depend on u , h , and p .

Corollary 8.2. *Let $1 < a \leq 2$, while the numbers α_1, α_2 , and h_1 be the same as in Corollary 8.1. Then there is a number $p_1 > 0$, independent of h , such that the operator $\mathbf{L}(p)$, $p \geq p_1$, $|h| \leq h_1$, has a bounded inverse and*

$$c_1 \|\mathbf{L}(p)u\|_{\mathcal{H}_a^0(Q, \Gamma)} \leq \|u\|_{H_a^2(Q)} \leq c_2 \|\mathbf{L}(p)u\|_{\mathcal{H}_a^0(Q, \Gamma)},$$

where $c_1, c_2 > 0$ do not depend on u , h , and p .

Remark 8.1. Let $\alpha_1 = \alpha_2$ and $|\alpha_1| < 1$. In this case, Corollaries 8.1 and 8.2 are true for any h_1 satisfying the relation $0 < h_1 < \pi/2$. Indeed, the matrix (8.21) remains positively definite because

$$e^{ih} + e^{-ih} = 2\mu > 0, \quad \det(e^{ih}R_1 + (e^{ih}R_1)^*) = 4\mu^2(1 - \alpha_1^2) > 0,$$

where $\mu = \operatorname{Re} e^{ih} > 0$. Therefore, the form $b_{\mathcal{R}}$ remains to be a closed sectorial form on $L_2(\theta)$ with the domain $D(b_{\mathcal{R}}) = \dot{W}^1(\theta)$ and vertex $k_2 > 0$. Further consideration does not change.

4. Consider the unbounded operators $\mathbf{A}, \mathbf{A}^1 : H_a^0(Q) \rightarrow H_a^0(Q)$ given by

$$\mathbf{A}u = -\Delta u, \quad u \in D(\mathbf{A}) = \{u \in H_a^2(Q) : u|_{\Gamma_l} + B_l^1u + B_l^2u = 0\},$$

$$\mathbf{A}^1u = -\Delta u, \quad u \in D(\mathbf{A}^1) = \{u \in H_a^2(Q) : u|_{\Gamma_l} + B_l^1u = 0\}.$$

Corollary 8.1 implies the following result (see Fig. 8.1).

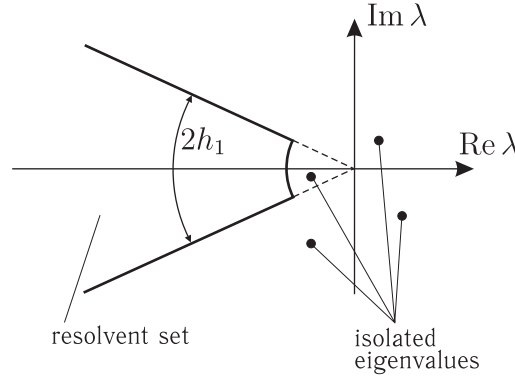


Figure 8.1: Spectra of the operators \mathbf{A}^1 and \mathbf{A}

Corollary 8.3. *Let the conditions of Corollary 8.1 hold. Then the following assertions are true:*

1. *the spectrum $\sigma(\mathbf{A}^1)$ is discrete⁵;*
2. *there exist numbers $h_1 = h_1(\alpha_1, \alpha_2) > 0$ and $\lambda_1 > 0$ such that $\sigma(\mathbf{A}^1) \subset \mathbb{C} \setminus \Omega_{h_1, \lambda_1}$, where*

$$\Omega_{h_1, \lambda_1} = \{\lambda \in \mathbb{C} : |\arg(\lambda - \pi)| \leq h_1, |\lambda| \geq \lambda_1\};$$

3. *the estimate*

$$\|(\mathbf{A}^1 - \lambda \mathbf{I})^{-1}\|_{H_a^0(Q) \rightarrow H_a^0(Q)} \leq c_1/|\lambda|, \quad \lambda \in \Omega_{h_1, \lambda_1},$$

holds with a constant $c_1 = c_1(h_1) > 0$.

⁵This means that the spectrum consists of a finite or a countable set of isolated eigenvalues of finite multiplicity.

Proof. Set $-\lambda = e^{ih}p^2$ and $\lambda_1 = p_0^2$, where p_0 is the constant occurring in Corollary 8.1. In this case, assertions 2 and 3 follow from Corollary 8.1. By the same corollary, the operator $(\mathbf{A} - \lambda\mathbf{I})^{-1} : H_a^0(Q) \rightarrow H_a^2(Q)$ is bounded for $\lambda \in \Omega_{h_1, \lambda_1}$. Combining this fact with the compactness of the embedding $H_a^2(Q) \subset H_a^0(Q)$, we see that the resolvent $(\mathbf{A} - \lambda\mathbf{I})^{-1} : H_a^0(Q) \rightarrow H_a^0(Q)$ is compact for $\lambda \in \Omega_{h_1, \lambda_1}$. Therefore, by Theorem 6.29 in [13, Chap. 3, Sec. 6], assertion 1 is true. \square

Using Corollary 8.2 instead of Corollary 8.1, we obtain the following result (see Fig. 8.1).

Corollary 8.4. *Let the conditions of Corollary 8.2 hold. Then the following assertions are true:*

1. *the spectrum $\sigma(\mathbf{A})$ is discrete;*
2. *there exists a number $\lambda_2 > 0$ such that $\sigma(\mathbf{A}) \subset \mathbb{C} \setminus \Omega_{h_1, \lambda_2}$;*
3. *the estimate*

$$\|(\mathbf{A} - \lambda\mathbf{I})^{-1}\|_{H_a^0(Q) \rightarrow H_a^0(Q)} \leq c_2/|\lambda|, \quad \lambda \in \Omega_{h_1, \lambda_2},$$

holds with a constant $c_2 = c_2(h_1) > 0$, where $h_1 > 0$ is the constant from Corollary 8.3.

Remark 8.2. Let $\alpha_1 = \alpha_2$ and $|\alpha_1| < 1$. In this case, Corollaries 8.3 and 8.4 are true for any h_1 satisfying the relation $0 < h_1 < \pi/2$ (cf. Remark 8.1).

The following questions are unanswered. Do there exist a number h_1 , $\pi/2 \leq h_1 < \pi$, and numbers $\lambda_1, \lambda_2 > 0$ such that

$$\sigma(\mathbf{A}^1) \subset \mathbb{C} \setminus \Omega_{h_1, \lambda_1}, \quad \sigma(\mathbf{A}) \subset \mathbb{C} \setminus \Omega_{h_1, \lambda_2}? \quad (8.25)$$

Can one find, for any h_1 , $0 < h_1 < \pi$, numbers $\lambda_1, \lambda_2 > 0$ such that relations (8.25) hold (cf. Problem 13.1 in [28, Sec. 13])?

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